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Weakly nonlocal symplectic structures, Whitham method and weakly nonlocal symplectic structures of hydrodynamic type

A Ya Maltsev

L D Landau Institute for Theoretical Physics, 119334 ul. Kosygina 2, Moscow, Russia

E-mail: maltsev@itp.ac.ru

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Abstract
We consider the special type of field-theoretical symplectic structures called weakly nonlocal. The structures of this type are, in particular, very common for integrable systems such as KdV or NLS. We introduce here the special class of weakly nonlocal symplectic structures which we call weakly nonlocal symplectic structures of hydrodynamic type. We investigate then the connection of such structures with the Whitham averaging method and propose the procedure of 'averaging' the weakly nonlocal symplectic structures. The averaging procedure gives the weakly nonlocal symplectic structure of hydrodynamic type for the corresponding Whitham system. The procedure also gives 'action variables' corresponding to the wave numbers of m-phase solutions of the initial system which give the additional conservation laws for the Whitham system.

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1. Introduction

We consider weakly nonlocal symplectic structures having the form

$$\Omega_{ij}(x,y) = \sum_{k\geqslant 0} \omega_{ij}^{(k)}(\varphi,\varphi_x,\ldots)\delta^{(k)}(x-y) + \sum_{s=1}^{g} e_s q_i^{(s)}(\varphi,\varphi_x,\ldots)\nu(x-y)q_j^{(s)}(\varphi,\varphi_y,\ldots). \tag{1.1}$$

We put here $\varphi = (\varphi^1,\ldots,\varphi^n)$, $i, j = 1,\ldots,n$, $e_s = \pm 1$, $\nu(x-y) = \frac{1}{2}\,\mathrm{sign}\,(x-y)$ and $\omega_{ij}^{(k)}$ and $q_i^{(s)}$ are some local functions of φ and its derivatives at the same point. We assume that both sums contain a finite number of terms and all $\omega_{ij}^{(k)}$ and $q_i^{(s)}$ depend on a finite number of derivatives of φ.

The form (1.1) can also be written in a more general form

$$\Omega_{ij}(x, y) = \sum_{k\geqslant 0} \omega^{(k)}_{ij}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x - y) + \sum_{s,p=1}^{g} \kappa_{sp} q^{(s)}_i(\varphi, \varphi_x, \ldots)\nu(x - y)q^{(p)}_j(\varphi, \varphi_y, \ldots)$$

where κ_{sp} is some constant symmetric bilinear form. The form (1.1) gives then the 'diagonal' representation of the nonlocal part in the appropriate basis $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(g)}$.

The form (1.1) will play the role of the 'symplectic' 2-form on the space of functions

$$\varphi(x) = (\varphi^1(x), \ldots, \varphi^n(x)), \qquad -\infty < x < +\infty$$

with the appropriate behaviour at infinity. We will put for simplicity $\varphi^i(x) \to 0$ or, more generally, $\varphi^i(x) \to$ const for $x \to \pm\infty$ in this paper. Let us call the corresponding space the loop space $\mathcal{L}_0$. We require that expression (1.1) gives the skew-symmetric closed 2-form on the space $\mathcal{L}_0$ (let us not put here the requirement of non-degeneracy).

The weakly nonlocal symplectic structures (1.1) were introduced in [9] where the fact that the 'negative' symplectic structures for KdV and NLS have this form was also proved.

Let us also state here a few words about the weakly nonlocal structures in the theory of integrable systems. Namely, we mention the weakly nonlocal Hamiltonian and symplectic structures which seem to be closely connected with local PDEs integrable in the sense of the inverse scattering method. We will call here (as in [9]) the Hamiltonian structure on $\mathcal{L}_0$ weakly nonlocal if it has a form similar to (1.1), i.e. the Poisson brackets of fields $\varphi^i(x)$ and $\varphi^j(y)$ can be formally written as

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k\geqslant 0} B^{ij}_{(k)}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x - y) + \sum_{s=1}^{g} e_s S^i_{(s)}(\varphi, \varphi_x, \ldots)\nu(x - y)S^j_{(s)}(\varphi, \varphi_y, \ldots) \tag{1.2}$$

with $e_s = \pm 1$.

We can also introduce the Hamiltonian operator $\hat{J}^{ij}$,

$$\hat{J}^{ij} = \sum_{k\geqslant 0} B^{ij}_{(k)}(\varphi, \varphi_x, \ldots)\frac{\partial^k}{\partial x^k} + \sum_{s=1}^{g} e_s S^i_{(s)}(\varphi, \varphi_x, \ldots)D^{-1}S^j_{(s)}(\varphi, \varphi_x, \ldots) \tag{1.3}$$

where D^{-1} is the integration operator defined in the skew-symmetric way:

$$D^{-1}\xi(x) = \frac{1}{2}\int_{-\infty}^{x} \xi(y)\,\mathrm{d}y - \frac{1}{2}\int_{x}^{+\infty} \xi(y)\,\mathrm{d}y.$$

For the functional $H[\varphi]$ the corresponding dynamical system can be written in the form

$$\varphi^i_t = \hat{J}^{ij}\frac{\delta H}{\delta\varphi^j(x)} = \sum_{k\geqslant 0} B^{ij}_{(k)}(\varphi, \varphi_x, \ldots)\frac{\partial^k}{\partial x^k}\frac{\delta H}{\delta\varphi^j(x)} + \sum_{s=1}^{g} e_s S^i_{(s)}(\varphi, \varphi_x, \ldots)D^{-1}\left[S^j_{(s)}(\varphi, \varphi_x, \ldots)\frac{\delta H}{\delta\varphi^j(x)}\right]. \tag{1.4}$$

The operator (1.3) should also be skew-symmetric and satisfy the Jacobi identity,

$$\frac{\delta J^{ij}(x, y)}{\delta\varphi^k(z)} + \frac{\delta J^{jk}(y, z)}{\delta\varphi^i(x)} + \frac{\delta J^{ki}(z, x)}{\delta\varphi^j(y)} \equiv 0$$

(in the sense of distributions).

It is not difficult to see that the functional

$$H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots)\, \mathrm{d}x$$

generates a local dynamical system

$$\varphi^i_t = S^i(\varphi, \varphi_x, \ldots)$$

according to (1.4) if it gives a conservation law for all the dynamical systems

$$\varphi^i_{t_s} = S^i_{(s)}(\varphi, \varphi_x, \ldots), \tag{1.5}$$

that is,

$$h_{t_s} \equiv \partial_x Q_s(\varphi, \varphi_x, \ldots)$$

for some functions $Q_s(\varphi, \varphi_x, \ldots)$.

As far as we know the first example of the Poisson bracket in this form (actually with zero local part) was the Sokolov bracket [5]

$$\{\varphi(x), \varphi(y)\} = \varphi_x \nu(x - y)\varphi_y$$

for the Krichever–Novikov equation [6]

$$\varphi_t = \varphi_{xxx} - \frac{3}{2}\frac{\varphi_{xx}{}^2}{\varphi_x} + \frac{h(\varphi)}{\varphi_x} = \varphi_x D^{-1}\varphi_x \frac{\delta H}{\delta\varphi(x)}$$

where $h(\varphi) = c_3\varphi^3 + c_2\varphi^2 + c_1\varphi + c_0$ and

$$H = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\frac{\varphi_{xx}{}^2}{\varphi_x{}^2} + \frac{1}{3}\frac{h(\varphi)}{\varphi_x{}^2} \right) \mathrm{d}x.$$

This equation appeared originally in [6] describing the ‘rank 2’ solutions of the KP system. In pure algebra, it describes the deformations of the commuting genus 1 pair OD operators of rank 2 whose classification was obtained in this work. As was found later, the Krichever–Novikov equation is a unique third order in x completely integrable evolution equation which cannot be reduced to KdV by Miura-type transformations.

The symplectic structure corresponding to the Sokolov bracket is purely local:

$$\Omega(x, y) = \frac{1}{\varphi_x}\delta'(x - y)\frac{1}{\varphi_y}.$$

Let us mention that the local symplectic structures were considered by Dorfman and Mokhov (see review [7]).

The hierarchy of the Poisson structures having the general form (1.2) was first written in [8] for KdV,

$$\varphi_t = 6\varphi\varphi_x - \varphi_{xxx},$$

using the local bi-Hamiltonian formalism (Gardner–Zakharov–Faddeev and Magri brackets) and the corresponding recursion operator in the Lenard–Magri scheme. Let us present here the pair of corresponding local Hamiltonian structures

$$\hat{J}_0 = \partial/\partial x$$

(Gardner–Zakharov–Faddeev bracket) and

$$\hat{J}_1 = -\partial^3/\partial x^3 + 2(\varphi\partial/\partial x + \partial/\partial x\varphi)$$

(Magri bracket) and the first weakly nonlocal Hamiltonian operator

$$\begin{aligned}\hat{J}_2 = {} & \partial^5/\partial x^5 - 8\varphi\partial^3/\partial x^3 - 12\varphi_x\partial^2/\partial x^2 - 8\varphi_{xx}\partial/\partial x \\ & + 16\varphi^2\partial/\partial x - 2\varphi_{xxx} + 16\varphi\varphi_x - 4\varphi_x D^{-1}\varphi_x.\end{aligned}$$

The operator $\hat{J}_2$ is obtained by the action of the recursion operator

$$\hat{R} = -\partial^2/\partial x^2 + 4\varphi + 2\varphi_x D^{-1}$$

(such that $\hat{R}\hat{J}_0 = \hat{J}_1$) on the operator $\hat{J}_1$. The higher ('positive') Hamiltonian operators $\hat{J}_n$ can be obtained in the same recursion scheme by the formula $\hat{J}_n = \hat{R}^n \hat{J}_0$. It was proved in [8] that all operators $\hat{J}_n$ for $n > 1$ can be written in the form

$$\hat{J}_n = (\text{local part}) - \sum_{k=1}^{n-1} S_{(k)}(\varphi, \varphi_x, \ldots) D^{-1} S_{(n-k-1)}(\varphi, \varphi_x, \ldots)$$

where $S_{(1)}(\varphi, \varphi_x, \ldots) = 2\varphi_x$ and

$$S_{(k)}(\varphi, \varphi_x, \ldots) \equiv \hat{R} S_{(k-1)}(\varphi, \varphi_x, \ldots)$$

are higher KdV flows.

The similar weakly nonlocal expressions for positive powers of the recursion operator for KdV were also considered in [8]. Let us present here the corresponding result,

$$\hat{R}^n = (\text{local part}) + \sum_{k=1}^{n} S_{(k)}(\varphi, \varphi_x, \ldots) D^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)}, \qquad n \geqslant 0$$

where $S_{(k)} = \partial_x \delta H_{(k)}/\delta\varphi(x)$, $H_{(0)} = \int \varphi \, \mathrm{d}x$ and

$$\frac{\delta H_{(k)}}{\delta \varphi(x)} \equiv \frac{\delta H_{(k-1)}}{\delta \varphi(x)} \hat{R}$$

are Euler–Lagrange derivatives of higher Hamiltonian functions for the KdV hierarchy. Let us also mention that in our notation $\hat{R}$ acts from the left on the vectors and from the right on the 1-forms in the functional space $\mathcal{L}_0$.

Using the results of [8] it was proved in [9] that the 'negative' symplectic structures (i.e. the inverse of 'negative' Hamiltonian operators) also have the weakly nonlocal form. Let us formulate here the corresponding statement.

All the 'negative' symplectic structures $\hat{\Omega}_{-n} = (\hat{J}_{-n})^{-1}$, $n \geqslant 0$ for the KdV hierarchy can be written in the following form:

$$\Omega_{-n} = (\text{local part}) + \sum_{k=0}^{n} \frac{\delta H_{(k)}}{\delta \varphi(x)} D^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)}.$$

It was conjectured in [9] that this structure of 'positive' Hamiltonian and 'negative' symplectic hierarchies should be very common for the wide class of integrable systems. In particular, similar statements about the NLS equation

$$\mathrm{i}\psi_t = -\psi_{xx} + 2\kappa|\psi|^2\psi$$

were proved in [9]. Let us give here also the corresponding statements for this case.

Two basic Hamiltonian operators can be written here in the form

$$\hat{J}_0 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \hat{J}_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} - 2\kappa \begin{pmatrix} -\psi\partial^{-1}\psi & \psi\partial^{-1}\bar{\psi} \\ \bar{\psi}\partial^{-1}\psi & -\bar{\psi}\partial^{-1}\bar{\psi} \end{pmatrix}.$$

The recursion operator $\hat{R}$ is defined again by formula $\hat{R}\hat{J}_0 = \hat{J}_1$. For the 'positive' Hamiltonian operators $\hat{J}_n = \hat{R}^n \hat{J}_0$ and 'negative' symplectic structures $\hat{\Omega}_{-n} = (\hat{J}_{-n})^{-1}$, $n \geqslant 1$ the following statements will then be true [9].

The 'positive' Hamiltonian operators $\hat{J}_n$ and 'negative' symplectic structures $\hat{\Omega}_{-n}$ in the hierarchy of Hamiltonian structures for NLS can be written in the form

$$\hat{J}_n = (\text{local part}) - \sum_{k=1}^{n} S_{(k-1)}(\psi, \bar{\psi}, \ldots) D^{-1} S_{(n-k)}(\psi, \bar{\psi}, \ldots)$$

$$\hat{\Omega}_{-n} = (\text{local part}) + \sum_{k=1}^{n} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} D^{-1} \frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)}$$

where

$$S_{(k)} \equiv \hat{J}_0 \frac{\delta H_{(k)}}{\delta(\psi, \bar{\psi})(x)}, \qquad H_{(0)} = \sqrt{2\kappa} \int \psi \bar{\psi}\, \mathrm{d}x, \qquad \text{and} \qquad \frac{\delta H_{(k)}}{\delta(\psi, \bar{\psi})(x)} = \hat{R} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)}$$

for any $k \geqslant 1$.[1]

General investigations of the weakly nonlocal structures of integrable hierarchies were made in very recent works. Let us cite here the work [11] (see also references therein) where the weakly nonlocal form of the structures described above was established for the integrable hierarchies under rather general requirements.

It is possible to state that the weakly nonlocal structures play indeed quite an important role in the theory of integrable systems.

Let us state that the 'positive' symplectic structures $\hat{\Omega}_n = \hat{J}_n^{-1}$ and the 'negative' Hamiltonian operators $\hat{J}_{-n}$ $(n \geqslant 1)$ will have much more complicated form (not weakly nonlocal) both for KdV and NLS hierarchies.

Let us formulate the theorem proved in [29] connecting the nonlocal and local parts for the general weakly nonlocal Poisson brackets (1.2). We will assume that the bracket (1.2) is written in 'irreducible' form, i.e. the 'vector fields',

$$\mathbf{S}_{(s)}(\varphi, \varphi_x, \ldots) = \left(S^1_{(s)}(\varphi, \varphi_x, \ldots), \ldots S^n_{(s)}(\varphi, \varphi_x, \ldots)\right)^t$$

which are linearly independent (with constant coefficients).

Theorem. *For any bracket (1.2) the flows*

$$\varphi^i_{t_s} = S^i_{(s)}(\varphi, \varphi_x, \ldots) \tag{1.6}$$

commute with each other and leave the bracket (1.2) invariant.

The second statement means here that the Lie derivative of the tensor (1.2) along the flows (1.6) is zero on the functional space $\mathcal{L}_0$.

However, the general classification of weakly nonlocal Hamiltonian structures (1.2) is rather difficult and is unavailable at present.

Let us state now a few words about a very important class of weakly nonlocal Hamiltonian and symplectic structures of hydrodynamic type (HT). These structures are closely connected with the systems of hydrodynamic type (HT systems), i.e. the systems of the form

$$U^\nu_T = V^\nu_\mu(\mathbf{U}) U^\mu_X, \qquad \nu, \mu = 1, \ldots, N \tag{1.7}$$

where $V^\nu_\mu(U)$ is some $N \times N$ matrix depending on the variables $U^1, \ldots, U^N$.

The Hamiltonian approach to systems (1.7) was started by Dubrovin and Novikov [16, 19, 21] who introduced the local (homogeneous) Poisson brackets of hydrodynamic type (Dubrovin–Novikov brackets). Let us give here the corresponding definition.

[1] Actually, as was pointed out in [9] the NLS equation has in fact three local Hamiltonian structures ($\hat{J}_0$, $\hat{J}_1$, $\hat{J}_2$) in the variables $r = \sqrt{\psi\bar{\psi}}$, $\theta = -\mathrm{i}(\psi_x/\psi - \bar{\psi}_x/\bar{\psi})$ (i.e. $\psi = r \exp(\mathrm{i} \int \theta\, \mathrm{d}x)$).

Definition 1. *A Dubrovin–Novikov bracket (DN bracket) is a bracket on the functional space* $(U^1(X), \dots, U^N(X))$ *having the form*

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X-Y) + b^{\nu\mu}_\lambda(\mathbf{U})U^\lambda_X \delta(X-Y) \tag{1.8}$$

The corresponding Hamiltonian operator $\hat{J}^{\nu\mu}$ can be written as

$$\hat{J}^{\nu\mu} = g^{\nu\mu}(\mathbf{U})\frac{\partial}{\partial x} + b^{\nu\mu}_\lambda(\mathbf{U})U^\lambda_X$$

and is homogeneous w.r.t. transformation $X \to aX$.

Every functional H of hydrodynamic type, i.e. the functional having the form

$$H = \int_{-\infty}^{+\infty} h(\mathbf{U})\,\mathrm{d}X$$

generates a system of hydrodynamic type (1.7) according to the formula

$$U^\nu_T = \hat{J}^{\nu\mu}\frac{\delta H}{\delta U^\mu(X)} = g^{\nu\mu}(\mathbf{U})\frac{\partial}{\partial x}\frac{\partial h}{\partial U^\mu} + b^{\nu\mu}_\lambda(\mathbf{U})\frac{\partial h}{\partial U^\mu}U^\lambda_X. \tag{1.9}$$

The DN bracket (1.8) is called non-degenerate if $\det\|g^{\nu\mu}(\mathbf{U})\| \neq 0$.

As was shown by Dubrovin and Novikov, the theory of DN brackets is closely connected with Riemannian geometry [16, 19, 21]. In fact, it follows from the skew-symmetry of (1.8) that the coefficients $g^{\nu\mu}(\mathbf{U})$ give, in the non-degenerate case, the contravariant pseudo-Riemannian metric on the manifold $\mathcal{M}^N$ with coordinates $(U^1, \dots, U^N)$ while the functions $\Gamma^\nu_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_\lambda(\mathbf{U})$ (where $g_{\nu\mu}(\mathbf{U})$ is the corresponding metric with lower indices) give the connection coefficients compatible with metric $g_{\nu\mu}(\mathbf{U})$. The validity of the Jacobi identity requires then that $g_{\nu\mu}(\mathbf{U})$ be actually a flat metric on the manifold $\mathcal{M}^N$ and the functions $\Gamma^\nu_{\mu\lambda}(\mathbf{U})$ give a symmetric (Levi-Civita) connection on $\mathcal{M}^N$ [16, 19, 21].

In the flat coordinates $n^1(\mathbf{U}), \dots, n^N(\mathbf{U})$ the non-degenerate DN bracket can be written in the constant form

$$\{n^\nu(X), n^\mu(Y)\} = \mathrm{e}^\nu \delta^{\nu\mu}\delta'(X-Y)$$

where $\mathrm{e}^\nu = \pm 1$.

The functionals

$$N^\nu = \int_{-\infty}^{+\infty} n^\nu(X)\,\mathrm{d}X$$

are the annihilators of the bracket (1.8) and the functional

$$P = \frac{1}{2}\int_{-\infty}^{+\infty}\sum_{\nu=1}^{N} \mathrm{e}^\nu\,(n^\nu(X))^2\,\mathrm{d}X$$

is the momentum functional generating the system $U^\nu_T = U^\nu_X$ according to (1.9).

The symplectic structure corresponding to a non-degenerate DN bracket has the weakly nonlocal form and can be written as

$$\Omega_{\nu\mu}(X, Y) = \mathrm{e}^\nu \delta_{\nu\mu}\nu(X-Y)$$

in coordinates n^ν or, more generally,

$$\Omega_{\nu\mu}(X, Y) = \sum_{\lambda=1}^{N} \mathrm{e}^\lambda \frac{\partial n^\lambda}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial n^\lambda}{\partial U^\mu}(Y)$$

in arbitrary coordinates U^ν.

Let us also mention that the degenerate brackets (1.8) are more complicated but also have a nice differential geometric structure [23].

The brackets (1.8) are closely connected with the integration theory of systems of hydrodynamic type (1.7). Namely, according to the conjecture of Novikov, all the diagonalizable systems (1.7) which are Hamiltonian with respect to DN brackets (1.8) (with Hamiltonian function of hydrodynamic type) are completely integrable. This conjecture was proved by Tsarev [41] who proposed a general procedure ('generalized hodograph method') of integration of Hamiltonian diagonalizable systems (1.7).

In fact, Tsarev's 'generalized hodograph method' permits us to integrate the wider class of diagonalizable systems (1.7) (semi-Hamiltonian systems [41]) which appeared to be Hamiltonian in a more general (weakly nonlocal) Hamiltonian formalism.

The corresponding Poisson brackets (Mokhov–Ferapontov bracket and Ferapontov bracket) are the weakly nonlocal generalizations of the DN bracket (1.8) and are connected with the geometry of submanifolds in pseudo-Euclidean spaces. Let us describe here the corresponding structures.

The Mokhov–Ferapontov bracket (MF bracket) has the form [42]

$$\{U^{\nu}(X), U^{\mu}(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X-Y) + b^{\nu\mu}_{\lambda}(\mathbf{U})U^{\lambda}_{X}\delta(X-Y) + cU^{\nu}_{X}\nu(X-Y)U^{\mu}_{Y}. \tag{1.10}$$

As was proved in [42], expression (1.10) with $\det\|g^{\nu\mu}(\mathbf{U})\| \neq 0$ gives the Poisson bracket on the space $U^{\nu}(X)$ if and only if

(1) the tensor $g^{\nu\mu}(\mathbf{U})$ represents the pseudo-Riemannian contravariant metric of constant curvature c on the manifold $\mathcal{M}^N$, i.e.,

$$R^{\nu\mu}_{\lambda\eta}(\mathbf{U}) = c\big(\delta^{\nu}_{\lambda}\delta^{\mu}_{\eta} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\eta}\big);$$

(2) the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_{\lambda}(\mathbf{U})$ represent the Levi-Civita connection of metric $g_{\nu\mu}(\mathbf{U})$.

The Ferapontov bracket (F bracket) is a more general weakly nonlocal generalization of the DN bracket having the form [43–46],

$$\{U^{\nu}(X), U^{\mu}(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X-Y) + b^{\nu\mu}_{\lambda}(\mathbf{U})U^{\lambda}_{X}\delta(X-Y) + \sum_{k=1}^{g} e_k w^{\nu}_{(k)\lambda}(\mathbf{U})U^{\lambda}_{X}\nu(X-Y)w^{\mu}_{(k)\delta}(\mathbf{U})U^{\delta}_{Y} \tag{1.11}$$

$e_k = \pm 1$, $\nu, \mu = 1, \ldots, N$.

Expression (1.11) (with $\det\|g^{\nu\mu}(\mathbf{U})\| \neq 0$) gives the Poisson bracket on the space $U^{\nu}(X)$ if and only if [43, 46]

(1) tensor $g^{\nu\mu}(\mathbf{U})$ represents the metric of the submanifold $\mathcal{M}^N \subset \mathbb{E}^{N+g}$ with flat normal connection in the pseudo-Euclidean space $\mathbb{E}^{N+g}$ of dimension $N+g$;

(2) the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_{\lambda}(\mathbf{U})$ represent the Levi-Civita connection of metric $g_{\nu\mu}(\mathbf{U})$;

(3) the set of affinors $\big\{w^{\nu}_{(k)\lambda}(\mathbf{U})\big\}$ represents the full set of Weingarten operators corresponding to g linearly independent parallel vector fields in the normal bundle, such that

$$g_{\nu\tau}(\mathbf{U})w^{\tau}_{(k)\mu}(\mathbf{U}) = g_{\mu\tau}(\mathbf{U})w^{\tau}_{(k)\nu}(\mathbf{U}), \qquad \nabla_{\nu}w^{\mu}_{(k)\lambda}(\mathbf{U}) = \nabla_{\lambda}w^{\mu}_{(k)\nu}(\mathbf{U})$$

$$R^{\nu\mu}_{\lambda\eta}(\mathbf{U}) = \sum_{k=1}^{g} e_k\big(w^{\nu}_{(k)\lambda}(\mathbf{U})w^{\mu}_{(k)\eta}(\mathbf{U}) - w^{\mu}_{(k)\lambda}(\mathbf{U})w^{\nu}_{(k)\eta}(\mathbf{U})\big).$$

Besides that the set of affinors $w_{(k)}$ is commutative $[w_{(k)}, w_{(k')}] = 0$.

As was shown in [44] expression (1.11) can be considered the Dirac reduction of the Dubrovin–Novikov bracket connected with the metric in $\mathbb{E}^{N+g}$ to the manifold $\mathcal{M}^N$ with flat normal connection. Let us also note that the MF bracket can be considered as a case of the F bracket when $\mathcal{M}^N$ is a (pseudo)-sphere $\mathcal{S}^N \subset \mathbb{E}^{N+1}$ in a pseudo-Euclidean space.

The symplectic structures $\Omega_{\nu\mu}(X, Y)$ for both (non-degenerate) MF bracket and F bracket also have the weakly nonlocal form [9, 10] and can be written in general coordinates U^ν as

$$\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{N+g} \epsilon_s \frac{\partial n^s}{\partial U^\nu}(X) \nu(X - Y) \frac{\partial n^s}{\partial U^\mu}(Y)$$

where $\epsilon_s = \pm 1$ and the metric G_{IJ} in the space $\mathbb{E}^{N+g}$ has the form $G_{IJ} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{N+g})$. The functions $n^1(\mathbf{U}), \ldots, n^{N+g}(\mathbf{U})$ are the 'canonical forms' on the manifold $\mathcal{M}^N$ and play the role of densities and annihilators of bracket (1.11) and 'canonical Hamiltonian functions' (see [9]) depending on the definition of phase space. In fact, the functions $n^s(\mathbf{U})$ are the restrictions of flat coordinates of metric G_{IJ} giving the DN bracket in $\mathbb{E}^{N+g}$ on manifold $\mathcal{M}^N$. The mapping $\mathcal{M}^N \to \mathbb{E}^{N+g}$,

$$(U^1, \ldots, U^N) \to (n^1(\mathbf{U}), \ldots, n^{N+g}(\mathbf{U}))$$

gives locally the embedding of $\mathcal{M}^N$ in $\mathbb{E}^{N+g}$ as a submanifold with flat normal connection.

All the brackets (1.8), (1.10), (1.11) are connected with Tsarev's method of integration of systems (1.7). Namely, any diagonalizable system (1.7) Hamiltonian w.r.t. the (non-degenerate) bracket (1.8), (1.10) or (1.11) can be integrated by the 'generalized hodograph method'.

We will not describe here Tsarev's method in detail. However, let us point out that the 'generalized hodograph method' and the HT Hamiltonian structures were very useful for Whitham's systems obtained by averaging of integrable PDEs [13, 15–21].

Let us now discuss the Whitham averaging method [13, 15–22]. We will restrict ourselves to the evolution systems

$$\varphi^i_t = Q^i(\varphi, \varphi_x, \ldots) \tag{1.12}$$

although the Whitham method can also be applied to more general PDE systems.

The m-phase Whitham averaging method is based on the existence of the finite-parametric family of solutions of (1.12) having the form

$$\varphi^i(x, t) = \Phi^i(\mathbf{k}(\mathbf{U})x + \boldsymbol{\omega}(\mathbf{U})t + \boldsymbol{\theta}_0, U^1, \ldots, U^N) \tag{1.13}$$

where $\mathbf{k} = (k^1, \ldots, k^m), \boldsymbol{\omega} = (\omega^1, \ldots, \omega^m), \boldsymbol{\theta} = (\theta^1, \ldots, \theta^m)$, and $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ are the functions 2π-periodic w.r.t. each θ^α and depending on the finite set of additional parameters $U^1, \ldots, U^N$. The solutions (1.13) are the quasiperiodic functions depending on $N + m$ parameters $U^1, \ldots, U^N$ and $\theta^1_0, \ldots, \theta^m_0$.

In the Whitham method the parameters $U^1, \ldots, U^N$ and $\theta^1_0, \ldots, \theta^m_0$ become the slow-modulated functions of x and t to get the slow-modulated m-phase solution of (1.12). We then introduce the slow variables $X = \epsilon x, T = \epsilon t, \epsilon \to 0$ and try to find a solution of the system

$$\epsilon \varphi^i_T = Q^i(\varphi, \epsilon \varphi_X, \ldots) \tag{1.14}$$

having the form

$$\varphi^i(X, T) = \sum_{k=0}^{+\infty} \epsilon^k \Phi^i_{(k)} \left(\frac{\mathbf{S}(X, T)}{\epsilon} + \boldsymbol{\theta}, X, T \right) \tag{1.15}$$

where $\Phi^i_{(k)}(\boldsymbol{\theta}, X, T)$ are 2π-periodic w.r.t. each θ^α and $\mathbf{S}(X, T) = (S^1(X, T), \ldots, S^m(X, T))$ is a 'phase' depending on the slow variables X and T [13, 14, 22].

It follows then that $\Phi^i_{(0)}(\boldsymbol{\theta}, X, T)$ should always belong to the family of exact m-phase solutions of (1.12) at any X and T and we have to find the functions $\Phi^i_{(k)}(\boldsymbol{\theta}, X, T), k \geqslant 1$ from the system (1.14). The existence of the solution (1.15) implies some conditions on the parameters $\mathbf{U}(X, T), \boldsymbol{\theta}_0(X, T)$ giving the zero approximation of (1.15). In particular, the existence of $\Phi^i_{(1)}(\boldsymbol{\theta}, X, T)$ implies the conditions on $\mathbf{U}(X, T)$ having the form of the system (1.7). This system is called the Whitham system and describes the evolution of the 'averaged' characteristics of the solution (1.15) in the main order. The solution of the Whitham system (1.7) is actually the main step in the whole procedure. Let us also mention that the Whitham systems for so-called 'integrable systems' like KdV can usually be written in the diagonal form [13, 15, 16, 19, 21, 48].

The Lagrangian formalism of the Whitham system and the averaging of the Lagrangian function were considered by Whitham [13] who pointed out that the Whitham system admits the (local) Lagrangian formalism if the initial system (1.12) was Lagrangian.

The Hamiltonian approach to the Whitham method was started by Dubrovin and Novikov in [16] (see also [19, 21]) where the procedure of 'averaging' the local field-theoretical Poisson bracket was proposed. The Dubrovin–Novikov procedure gives the DN bracket for the Whitham system (1.7) in the case when the initial system (1.12) is Hamiltonian w.r.t. a local Poisson bracket

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k\geqslant 0} B^{ij}_{(k)}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y)$$

with local Hamiltonian functional[2]

$$H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots)\,\mathrm{d}x.$$

This procedure was generalized in [28, 29] for the weakly nonlocal Hamiltonian structures. In this case, the procedure of construction of a general F bracket (or MF bracket) for the Whitham system from the weakly nonlocal Poisson bracket (1.2) for initial system (1.12) was proposed.

In this paper, we will consider the Whitham averaging method for PDEs having weakly nonlocal symplectic structures (1.1) and construct the symplectic structures of hydrodynamic type for the corresponding Whitham systems. Let us state that the corresponding HT symplectic structures can in principle be more general than those connected with the Tsarev integration method. The theory of integration of corresponding HT systems (1.7) should then be more complicated in the general case.

We call here the weakly nonlocal symplectic structure of hydrodynamic type the symplectic form $\Omega_{\nu\mu}(X, Y)$ having the form

$$\Omega_{\nu\mu}(X, Y) = \sum_{s,p=1}^{M} \kappa_{sp}\omega^{(s)}_\nu(\mathbf{U}(X))\nu(X-Y)\omega^{(p)}_\mu(\mathbf{U}(Y)) \tag{1.16}$$

or in the 'diagonal' form

$$\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{M} e_s\omega^{(s)}_\nu(\mathbf{U}(X))\nu(X-Y)\omega^{(s)}_\mu(\mathbf{U}(Y))$$

[2] The proof of Jacobi identity for the averaged bracket was obtained in [26].

in coordinates U^{ν} where κ_{sp} is some quadratic form, $e_s = \pm 1$ and $\omega^{(s)}_{\nu}(\mathbf{U})$ are closed 1-forms on the manifold $\mathcal{M}^N$. Locally, the forms $\omega^{(s)}_{\nu}(\mathbf{U})$ can be represented as the gradients of some functions $f^{(s)}(\mathbf{U})$ such that

$$\Omega_{\nu\mu}(X, Y) = \sum_{s,p=1}^{M} \kappa_{sp} \frac{\partial f^{(s)}}{\partial U^{\nu}}(X)\nu(X - Y)\frac{\partial f^{(p)}}{\partial U^{\mu}}(Y). \qquad (1.17)$$

Generally speaking, we do not require here that the embedding $\mathcal{M}^N \subset \mathbb{E}^M$ given by $(U^1, \ldots, U^N) \to (f^{(1)}(\mathbf{U}), \ldots, f^{(M)}(\mathbf{U}))$ gives the submanifold with flat normal connection. Therefore, the corresponding Hamiltonian operators will not necessarily have the weakly nonlocal form of the DN brackets, MF brackets or F brackets.

We propose here a procedure which permits us to construct the symplectic structure (1.16) for the Whitham system in the case when the (local) initial system (1.12) has weakly nonlocal symplectic structure (1.1) with some local Hamiltonian function

$$H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots)\,\mathrm{d}x.$$

In section 2 we consider the general symplectic forms (1.1) and the HT symplectic forms (1.16). In section 3 we consider the general features of the Whitham method and introduce some conditions which we will need for the next considerations. In section 4 we introduce the 'extended' phase space and prove some technical lemmas about the 'extended' symplectic form necessary for the averaging procedure of the forms (1.1). In section 5 we give the procedure for averaging the forms (1.1) and prove that the Whitham system admits the symplectic structure of hydrodynamic type given by the corresponding 'averaged' symplectic form. In section 6 we give another variant of averaging of forms (1.1) based on the averaging of weakly nonlocal 1-forms and give the weakly nonlocal Lagrangian formalism for the Whitham system.

2. General weakly nonlocal symplectic forms and the weakly nonlocal symplectic forms of hydrodynamic type

Let us consider first the general weakly nonlocal symplectic forms (1.1). The nonlocal part of (1.1) is skew-symmetric and we should require then also the skew-symmetry of the local part of (1.1). We will assume everywhere that (1.1) is written in 'irreducible' form, i.e. the functions $\mathbf{q}^{(s)}(\varphi, \varphi_x, \ldots)$ are linearly independent (with constant coefficients). Let us prove here the following statement formulated in [9].

Theorem 1. *For any closed 2-form (1.1) the functions $q^{(s)}_i(\varphi, \varphi_x, \ldots)$ represent the closed 1-forms on $\mathcal{L}_0$.*

Proof. Let us denote by $\Omega'_{ij}(x, y)$ the local part of (1.1). We have to check the closeness of 2-form (1.1), i.e.

$$(\mathrm{d}\Omega)_{ijk}(x, y, z) = \frac{\delta\Omega_{ij}(x, y)}{\delta\varphi^k(z)} + \frac{\delta\Omega_{jk}(y, z)}{\delta\varphi^i(x)} + \frac{\delta\Omega_{ki}(z, x)}{\delta\varphi^j(y)} \equiv 0$$

(in the sense of distributions) on $\mathcal{L}_0$.

We have then

$$(\mathrm{d}\Omega)_{ijk}(x, y, z) = (\mathrm{d}\Omega')_{ijk}(x, y, z) + \sum_{s=1}^{g} e_s \left[\frac{\delta q^{(s)}_i(x)}{\delta\varphi^k(z)}\nu(x - y)q^{(s)}_j(y) + q^{(s)}_i(x)\nu(x - y)\frac{\delta q^{(s)}_j(y)}{\delta\varphi^k(z)} \right]$$

$$+\sum_{s=1}^{g} e_s \left[\frac{\delta q_j^{(s)}(y)}{\delta \varphi^i(x)} \nu(y-z) q_k^{(s)}(z) + q_j^{(s)}(y) \nu(y-z) \frac{\delta q_k^{(s)}(z)}{\delta \varphi^i(x)} \right]$$
$$+\sum_{s=1}^{g} e_s \left[\frac{\delta q_k^{(s)}(z)}{\delta \varphi^j(y)} \nu(z-x) q_i^{(s)}(x) + q_k^{(s)}(z) \nu(z-x) \frac{\delta q_i^{(s)}(x)}{\delta \varphi^j(y)} \right]. \tag{2.1}$$

We use here the Leibnitz identity and the relations

$$\frac{\delta \varphi^i(x)}{\delta \varphi^j(y)} = \delta^i_j \delta(x-y), \qquad \frac{\delta \varphi^i_x(x)}{\delta \varphi^j(y)} = \delta^i_j \delta'(x-y), \ldots. \tag{2.2}$$

The expression $(\mathrm{d}\Omega')_{ijk}(x, y, z)$ is then purely local and all the nonlocality arises just in the remaining part of $(\mathrm{d}\Omega)_{ijk}(x, y, z)$. Let us consider now the values

$$\mathrm{d}\Omega(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\mathrm{d}\Omega)_{ijk}(x, y, z) \xi^i(x) \eta^j(y) \zeta^k(z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where $\xi^i(x)$, $\eta^i(x)$, $\zeta^i(x)$ are the functions with finite supports such that the supports of all $\zeta^k(x)$ do not intersect with the supports of all $\xi^i(x)$, $\eta^j(x)$ and moreover all supports of $\xi^i(x)$, $\eta^j(x)$ lie on the left of any support of $\zeta^k(x)$. Using (2.1) and (2.2) it is easy to see then that we can write in this case

$$\begin{aligned}
\mathrm{d}\Omega(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{s=1}^{g} e_s \\
& \times \left[\frac{\delta q_j^{(s)}(y)}{\delta \varphi^i(x)} \nu(y-z) q_k^{(s)}(z) + q_k^{(s)}(z) \nu(z-x) \frac{\delta q_i^{(s)}(x)}{\delta \varphi^j(y)} \right] \\
& \times \xi^i(x) \eta^j(y) \zeta^k(z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
= & \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{s=1}^{g} e_s \left[\frac{\delta q_i^{(s)}(x)}{\delta \varphi^j(y)} - \frac{\delta q_j^{(s)}(y)}{\delta \varphi^i(x)} \right] \\
& \times q_k^{(s)}(z) \xi^i(x) \eta^j(y) \zeta^k(z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
= & \frac{1}{2} \sum_{s=1}^{g} e_s \left[\int_{-\infty}^{+\infty} q_k^{(s)}(z) \zeta^k(z) \, \mathrm{d}z \right] \\
& \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{\delta q_i^{(s)}(x)}{\delta \varphi^j(y)} - \frac{\delta q_j^{(s)}(y)}{\delta \varphi^i(x)} \right] \xi^i(x) \eta^j(y) \, \mathrm{d}x \, \mathrm{d}y \equiv 0.
\end{aligned}$$

Let us now use the fact that the functions $q_i^{(s)}(x) = q_i^{(s)}(\varphi, \varphi_x, \ldots)$ are local translationally invariant (i.e. they do not depend explicitly on x) expressions depending on $\varphi(x)$ and their derivatives. Let us consider the functions $\varphi^i(x)$ which can be represented as

$$\varphi^i(x) = \tilde{\varphi}^i(x) + \tilde{\tilde{\varphi}}^i(x)$$

where

$$\mathrm{Supp}\, \tilde{\varphi}(x) \subset \bigcup_k \mathrm{Supp}\, \zeta^k(x) \qquad \mathrm{Supp}\, \tilde{\tilde{\varphi}}(x) \subset \left[\bigcup_i \mathrm{Supp}\, \xi^i(x) \right] \bigcup \left[\bigcup_j \mathrm{Supp}\, \eta^j(x) \right].$$

Denote

$$A^{(s)}[\tilde{\tilde{\varphi}}, \boldsymbol{\xi}, \boldsymbol{\eta}] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{\delta q_i^{(s)}(\tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}}_x, \ldots)}{\delta \tilde{\tilde{\varphi}}^j(y)} - \frac{\delta q_j^{(s)}(\tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}}_y, \ldots)}{\delta \tilde{\tilde{\varphi}}^j(x)} \right] \xi^i(x) \eta^j(x) \, \mathrm{d}x \, \mathrm{d}y.$$

We have then

$$\sum_{s=1}^{g} e_s A^{(s)}[\tilde{\boldsymbol{\varphi}}, \boldsymbol{\xi}, \boldsymbol{\eta}] \int_{-\infty}^{+\infty} q_k^{(s)}(\tilde{\varphi}, \tilde{\varphi}_z, \ldots)\zeta^k(z)\,\mathrm{d}z \equiv 0$$

(for all $\tilde{\varphi}(z)$, $\zeta^k(z)$).

It is easy to show that for linearly independent set $\mathbf{q}^{(s)}(\tilde{\varphi}, \tilde{\varphi}_z, \ldots)$ this system can have only trivial solution $A^{(s)}[\tilde{\boldsymbol{\varphi}}, \boldsymbol{\xi}, \boldsymbol{\eta}] \equiv 0$ for any $\xi^i(x), \eta^j(y)$ and $\tilde{\boldsymbol{\varphi}}$ which is equivalent to condition $(\mathrm{d}q^{(s)})_{ij}(x, y) = 0$ for any $q_i^{(s)}(\varphi, \varphi_x, \ldots)$. □

We will now put $q_i^{(s)}(\varphi, \varphi_x, \ldots) = \delta H^{(s)}/\delta\varphi^i(x)$ where $H^{(s)}$ are some 'local' functionals

$$H^{(s)}[\varphi] = \int_{-\infty}^{+\infty} h^{(s)}(\varphi, \varphi_x, \ldots)\,\mathrm{d}x$$

and $\delta/\delta\varphi^i(x)$ is the Euler–Lagrange derivative and consider the structures (1.1) in the form

$$\Omega_{ij}(x, y) = \sum_{k\geqslant 0} \omega_{ij}^{(k)}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y) + \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \nu(x-y) \frac{\delta H^{(s)}}{\delta\varphi^j(y)}. \tag{2.3}$$

Let us consider now the weakly nonlocal symplectic structures of hydrodynamic type (1.16).

Theorem 2. *Expression (1.16) gives the closed 2-form on the space* $\{\mathbf{U}(X)\}$ *if and only if the 1-forms* $\omega_\nu^{(s)}(\mathbf{U})$ *on* $\mathcal{M}^N$ *are closed*[3], *i.e.*

$$\frac{\partial}{\partial U^\nu}\omega_\mu^{(s)}(\mathbf{U}) = \frac{\partial}{\partial U^\mu}\omega_\nu^{(s)}(\mathbf{U}).$$

Let us say here that the statement analogous to theorem 2 was first proved by O I Mokhov for the weakly nonlocal symplectic operators of hydrodynamic type having the form $\hat{\omega}_{ij} = a_i(\mathbf{U})D^{-1}b_j(\mathbf{U}) + b_i(\mathbf{U})D^{-1}a_j(\mathbf{U})$ (see [38, 39]). Theorem 2 represents the not difficult generalization of this statement for the arbitrary number of terms in the non-local structure (1.16).

Proof. Let us use the 'diagonal' form of (1.16). It is easy to see that the form (1.16) is skew-symmetric. From theorem 1 we get that the forms $\omega_\nu^{(s)}(\mathbf{U})$ should be closed on the functional space $\{\mathbf{U}(X)\}$. We have then

$$\begin{aligned}\frac{\delta\omega_\mu^{(s)}(\mathbf{U}(Y))}{\delta U^\nu(X)} - \frac{\delta\omega_\nu^{(s)}(\mathbf{U}(X))}{\delta U^\mu(Y)} &= \frac{\partial\omega_\mu^{(s)}(\mathbf{U})}{\partial U^\nu}(Y)\delta(Y-X) - \frac{\partial\omega_\nu^{(s)}(\mathbf{U})}{\partial U^\mu}(X)\delta(X-Y)\\ &= \left[\frac{\partial\omega_\mu^{(s)}(\mathbf{U})}{\partial U^\nu}(X) - \frac{\partial\omega_\nu^{(s)}(\mathbf{U})}{\partial U^\mu}(X)\right]\delta(X-Y) \equiv 0.\end{aligned}$$

So we have

$$\frac{\partial\omega_\mu^{(s)}(\mathbf{U})}{\partial U^\nu} - \frac{\partial\omega_\nu^{(s)}(\mathbf{U})}{\partial U^\mu} \equiv 0.$$

[3] We assume that (1.16) is written in the 'irreducible' form, i.e. the 1-forms $\omega_\nu^{(s)}(\mathbf{U})$ are linearly independent (with constant coefficients).

It is not difficult now to get, by direct calculation, that $(\mathrm{d}\Omega)_{\nu\mu\lambda}(X, Y, Z)$ can be written in the form

$$
\begin{aligned}
(\mathrm{d}\Omega)_{\nu\mu\lambda}(X, Y, Z) = \sum_{s=1}^{M} e_s \omega_\nu^{(s)}(X)\nu(X - Y)\delta(Y - Z)\left[\frac{\partial\omega_\mu^{(s)}}{\partial U^\lambda}(Z) - \frac{\partial\omega_\lambda^{(s)}}{\partial U^\mu}(Z)\right] \\
+ \sum_{s=1}^{M} e_s \omega_\mu^{(s)}(Y)\nu(Y - Z)\delta(Z - X)\left[\frac{\partial\omega_\lambda^{(s)}}{\partial U^\nu}(X) - \frac{\partial\omega_\nu^{(s)}}{\partial U^\lambda}(X)\right] \\
+ \sum_{s=1}^{M} e_s \omega_\lambda^{(s)}(Z)\nu(Z - X)\delta(X - Y)\left[\frac{\partial\omega_\nu^{(s)}}{\partial U^\mu}(Y) - \frac{\partial\omega_\mu^{(s)}}{\partial U^\nu}(Y)\right].
\end{aligned}
$$

So we get the second part of the theorem. □

We can put locally $\omega_\nu^{(s)}(\mathbf{U}) = \partial f^{(s)}(\mathbf{U})/\partial U^\nu$ on $\mathcal{M}^N$ and write the symplectic structure (1.16) in a 'conservative form'

$$
\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{M} e_s \frac{\partial f^{(s)}}{\partial U^\nu}(X)\nu(X - Y)\frac{\partial f^{(s)}}{\partial U^\mu}(Y). \tag{2.4}
$$

We will usually consider the form $\Omega_{\nu\mu}(X, Y)$ on the loop space $\mathcal{L}_{P_0}$ such that $P_0 \in \mathcal{M}^N$ is some fixed point of $\mathcal{M}^N$ and the functions $\mathbf{U}(X) \to P_0$ (quickly enough) for $X \to \pm\infty$. The action of $\Omega_{\nu\mu}(X, Y)$ will be usually defined on the 'vector fields' $\xi^\nu(X)$ rapidly decreasing for $X \to \pm\infty$.

The 2-form $\Omega_{\nu\mu}(X, Y)$ written in the form (2.4) can be considered as the pullback of the form

$$
\Xi_{IJ}(X, Y) = e_I \delta_{IJ} \nu(X - Y), \qquad I, J = 1, \ldots, M
$$

defined in the pseudo-Euclidean space $\mathbb{E}^N$ with the metric $G_{IJ} = \mathrm{diag}(e_1, \ldots, e_M)$ for the mapping α: $\mathcal{M}^N \to \mathbb{E}^N$

$$
(U^1, \ldots, U^N) \to (f^{(1)}(\mathbf{U}), \ldots, f^{(M)}(\mathbf{U})).
$$

Definition 2. *We call the symplectic form (1.16) non-degenerate if $M \geqslant N$ and*

$$
\mathrm{rank}\begin{pmatrix} \omega_i^{(1)}(\mathbf{U}) \\ \cdots \\ \omega_i^{(M)}(\mathbf{U}) \end{pmatrix} = N.
$$

It is easy to see that the non-degeneracy of $\Omega_{\nu\mu}(X, Y)$ coincides with the condition of regularity of N-dimensional submanifold $\alpha(\mathcal{M}^N) \subset \mathbb{E}^N$ in the space $\mathbb{E}^N$ for $M \geqslant N$.

3. The families of *m*-phase solutions and the Whitham method

We will consider now the Whitham averaging method for the local systems

$$
\varphi_t^i = Q^i(\varphi, \varphi_x, \ldots) \tag{3.1}
$$

having the weakly nonlocal symplectic structure (2.3) with a 'local' Hamiltonian functional

$$
H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots)\, \mathrm{d}x. \tag{3.2}
$$

This means that

$$\int_{-\infty}^{+\infty} \Omega_{ij}(x, y)\varphi_t^j(y)\,\mathrm{d}y = \int_{-\infty}^{+\infty} \Omega_{ij}(x, y) Q^j(\varphi, \varphi_y, \ldots)\,\mathrm{d}y \equiv \frac{\delta H}{\delta\varphi^i(x)}$$

on $\hat{\mathcal{W}}_0$ where $\delta/\delta\varphi^i(x)$ is the Euler–Lagrange derivative.

This requires, in particular, that the functionals $H^{(s)}[\varphi]$ be the conservation laws for the system (3.1) such that

$$h_t^{(s)} \equiv \partial_x J^{(s)}(\varphi, \varphi_x, \ldots) \tag{3.3}$$

for some functions $J^{(s)}(\varphi, \varphi_x, \ldots)$. The functional $H[\varphi]$ is defined then actually up to the linear combination of $H^{(s)}[\varphi]$ depending on the boundary conditions at infinity.

We assume now that the system (3.1) has a finite-parametric family of quasiperiodic solutions

$$\varphi^i(x, t) = \Phi^i(\mathbf{k}(\mathbf{U})x + \boldsymbol{\omega}(\mathbf{U})t + \boldsymbol{\theta}_0, \mathbf{U}), \qquad i = 1, \ldots, n \tag{3.4}$$

where $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^m), \mathbf{k} = (k^1, \ldots, k^m), \boldsymbol{\omega} = (\omega^1, \ldots, \omega^m)$ and $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ give the family of functions 2π-periodic w.r.t. each θ^α depending on the additional parameters $\mathbf{U} = (U^1, \ldots, U^N)$.

The functions $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ satisfy the system

$$g^i(\boldsymbol{\Phi}, \omega^\alpha(\mathbf{U})\boldsymbol{\Phi}_{\theta^\alpha}, \ldots) = \omega^\alpha(\mathbf{U})\Phi^i_{\theta^\alpha} - Q^i(\boldsymbol{\Phi}, k^\alpha(\mathbf{U})\boldsymbol{\Phi}_{\theta^\alpha}, \ldots) = 0 \tag{3.5}$$

and we assume that the system (3.5) has the finite-parametric family Λ of solutions (for generic $\mathbf{k}$ and $\boldsymbol{\omega}$) on the space of functions 2π-periodic w.r.t. each θ^α with parameters $\mathbf{U} = (U^1, \ldots, U^N)$ and the 'initial phase shift' $\boldsymbol{\theta}_0 = \left(\theta_0^1, \ldots, \theta_0^m\right)$. We can choose then (in a smooth way) at every $(U^1, \ldots, U^N)$ some function $\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ as having zero initial phase shift and represent the m-phase solutions of system (3.1) in the form (3.4).

In the Whitham method we make a rescaling $X = \epsilon x$, $T = \epsilon t$ $(\epsilon \to 0)$ of both variables x and t and try to find a function

$$\mathbf{S}(X, T) = (S^1(X, T), \ldots, S^m(X, T)) \tag{3.6}$$

and 2π-periodic functions

$$\Psi^i(\boldsymbol{\theta}, X, T, \epsilon) = \sum_{k\geqslant 0} \Psi^i_{(k)}(\boldsymbol{\theta}, X, T)\epsilon^k \tag{3.7}$$

such that the functions

$$\phi^i(\boldsymbol{\theta}, X, T, \epsilon) = \Psi^i\left(\frac{\mathbf{S}(X, T)}{\epsilon} + \boldsymbol{\theta}, X, T, \epsilon\right) \tag{3.8}$$

satisfy the system

$$\epsilon\phi^i_T = Q^i(\phi, \epsilon\phi_X, \ldots) \tag{3.9}$$

at every X, T and $\boldsymbol{\theta}$.

It is easy to see that the function $\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ satisfies the system (3.5) at every X and T with

$$k^\alpha = S^\alpha_X, \qquad \omega^\alpha = S^\alpha_T$$

and so belongs at every (X, T) to the family Λ. We can then write

$$\Psi^i_{(0)}(\boldsymbol{\theta}, X, T) = \Phi^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T), \mathbf{U}(X, T)).$$

We can then introduce the functions $U^\nu(X, T)$, $\theta_0^\alpha(X, T)$ as the parameters characterizing the main term in (3.7) which should satisfy the condition

$$[k^\alpha(\mathbf{U})]_T = [\omega^\alpha(\mathbf{U})]_X. \tag{3.10}$$

We have to define now the functions $\Psi^i_{(1)}(\boldsymbol{\theta}, X, T)$ from the liner system

$$\hat{L}^i_j \Psi^j_{(1)}(\boldsymbol{\theta}, X, T) = f^i_{(1)}(\boldsymbol{\theta}, X, T) \tag{3.11}$$

where

$$\hat{L}^i_j = \hat{L}^i_{(X,T)j} = \delta^i_j \omega^\alpha(X,T)\frac{\partial}{\partial\theta^\alpha} - \frac{\partial Q^i}{\partial\varphi^j}(\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \ldots)$$
$$- \frac{\partial Q^i}{\partial\varphi^j_x}(\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \cdots) k^\alpha(X,T)\frac{\partial}{\partial\theta^\alpha} - \cdots \tag{3.12}$$

is the linearization of system (3.5) and $\mathbf{f}_{(1)}(\boldsymbol{\theta}, X, T)$ is the discrepancy given by

$$f^i_{(1)}(\boldsymbol{\theta}, X, T) = -\Psi^i_{(0)T}(\boldsymbol{\theta}, X, T) + \frac{\partial Q^i}{\partial\varphi^j_x}(\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \ldots)\Psi^j_{(0)X}(\boldsymbol{\theta}, X, T)$$
$$+ \frac{\partial Q^i}{\partial\varphi^j_{xx}}(\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \ldots)\big(2k^\alpha(X,T)\Psi^j_{(0)\theta^\alpha X} + k^\alpha_X \Psi^j_{(0)\theta^\alpha}\big) + \cdots \tag{3.13}$$

where

$$\frac{\partial}{\partial T} = U^\nu_T \frac{\partial}{\partial U^\nu} + \theta^\alpha_{(0)T}\frac{\partial}{\partial\theta^\alpha}, \qquad \frac{\partial}{\partial X} = U^\nu_X \frac{\partial}{\partial U^\nu} + \theta^\alpha_{(0)X}\frac{\partial}{\partial\theta^\alpha}$$

for the functions

$$\Psi^i_{(0)}(\boldsymbol{\theta}, X, T) = \Phi^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T)).$$

We will assume that k^α and ω^α can be considered (locally) as the independent parameters on the family Λ and the total family of solutions of (3.5) depends (for generic k^α, ω^α) on $N = 2m + r$ ($r \geqslant 0$) parameters U^ν and m initial phases $\theta^\alpha_{(0)}$.

It is easy to see that the functions $\boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$ and $\nabla_{\boldsymbol{\xi}}\boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$ where $\boldsymbol{\xi}$ is any vector in the space of parameters U^ν tangential to the surface $\mathbf{k} = \text{const}$, $\boldsymbol{\omega} = \text{const}$ belong to the kernel of operator $\hat{L}^i_{(X,T)j}$.

Let us put now some 'regularity' conditions on the family (3.4) of quasiperiodic solutions of (3.1).

Definition 3. *We call the family (3.4) the full regular family of m-phase solutions of (3.1) if*

(1) the functions $\boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{U})$, $\boldsymbol{\Phi}_{U^\nu}(\boldsymbol{\theta}, \mathbf{U})$ are linearly independent (almost everywhere) on the set Λ;

(2) the $m + r$ linearly independent functions $\boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{U})$, $\nabla_{\boldsymbol{\xi}}\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ ($\nabla_{\boldsymbol{\xi}}\mathbf{k} = 0, \nabla_{\boldsymbol{\xi}}\boldsymbol{\omega} = 0$) give the full kernel of the operator $\hat{L}^i_{[\mathbf{U}]j}$ (here $\boldsymbol{\theta}_0 = 0$) for generic $\mathbf{k}$ and $\boldsymbol{\omega}$;

(3) there are exactly $m+r$ linearly independent 'right eigenvectors' $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta})$, $q = 1, \ldots, m+r$ of the operator $\hat{L}^i_{[\mathbf{U}]j}$ (for generic $\mathbf{k}$ and $\boldsymbol{\omega}$) corresponding to zero eigenvalues, i.e.

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}]i}(\boldsymbol{\theta})\hat{L}^i_{[\mathbf{U}]j}\psi^j(\boldsymbol{\theta})\frac{\mathrm{d}^m\theta}{(2\pi)^m} \equiv 0$$

for any periodic $\psi^j(\boldsymbol{\theta})$.

We then have to put the $m+r$ conditions of orthogonality of the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta}, X, T)$ to the functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}](X,T)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X,T))$

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}(X,T)]i}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X,T)) f^i_{(1)}(\boldsymbol{\theta}, X, T)\frac{\mathrm{d}^m\theta}{(2\pi)^m} = 0 \tag{3.14}$$

at every X, T to be able to solve the system (3.11) on the space of functions periodic w.r.t. each θ^{α}.

The system (3.14) together with (3.10) gives $m+(m+r) = 2m+r = N$ conditions at each X and T on the parameters of zero approximation $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ necessary for the construction of the first ϵ-term in the solution (3.7). Let us prove now the following lemma about the orthogonality conditions (3.14):

Lemma 1. *Under all the assumptions of regularity formulated above the orthogonality conditions (3.14) do not contain the functions $\theta_0^{\alpha}(X, T)$ and give just the restriction on the functions $U^{\nu}(X, T)$ having the form*

$$C_{\nu}^{(q)}(\mathbf{U})U_T^{\nu} - D_{\nu}^{(q)}(\mathbf{U})U_X^{\nu} = 0$$

(with some functions $C_{\nu}^{(q)}(\mathbf{U})$, $D_{\nu}^{(q)}(\mathbf{U})$).

Proof. Let us write down the part $\tilde{\mathbf{f}}_{(1)}(\boldsymbol{\theta}, X, T)$ of $\mathbf{f}_{(1)}(\boldsymbol{\theta}, X, T)$ which contains the derivatives $\theta_{0T}^{\alpha}(X, T)$ and $\theta_{0X}^{\alpha}(X, T)$. We have from (3.13)

$$\tilde{f}_{(1)}^{i}(\boldsymbol{\theta}, X, T) = -\Psi_{(0)\theta^{\beta}}^{i}(\boldsymbol{\theta}, X, T)\theta_{0T}^{\beta} + \frac{\partial Q^i}{\partial \varphi_x^j}(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \ldots)\Psi_{(0)\theta^{\beta}}^{j}(\boldsymbol{\theta}, X, T)\theta_{0X}^{\beta}$$
$$+ \frac{\partial Q^i}{\partial \varphi_{xx}^j}(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \ldots)2k^{\alpha}(X, T)\Psi_{(0)\theta^{\alpha}\theta^{\beta}}^{j}(\boldsymbol{\theta}, X, T)\theta_{0X}^{\beta} + \cdots .$$

We can then write

$$\tilde{f}_{(1)}^{i}(\boldsymbol{\theta}, X, T) = \left[-\frac{\partial}{\partial\omega^{\beta}} g^i(\mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U}), \ldots) + \hat{L}^i_j \frac{\partial}{\partial\omega^{\beta}}(\Phi^j(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U}), \ldots)\right]\theta_{0T}^{\beta}$$
$$+ \left[\frac{\partial}{\partial k^{\beta}} g^i(\mathbf{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U}), \ldots) - \hat{L}^i_j \frac{\partial}{\partial k^{\beta}}(\Phi^j(\boldsymbol{\theta}+\boldsymbol{\theta}_0, \mathbf{U}), \ldots)\right]\theta_{0X}^{\beta}$$

where the constraints g^i and the operator $\hat{L}^i_{(X,T)j}$ were introduced in (3.5) and (3.12) respectively.

The derivatives $\partial g^i/\partial\omega^{\beta}$ and $\partial g^i/\partial k^{\beta}$ are identically zero on Λ according to (3.5). We have then

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \kappa_{[\mathbf{U}(X,T)]i}^{(q)}(\boldsymbol{\theta}+\boldsymbol{\theta}_0(X, T))\tilde{f}_{(1)}^{i}(\boldsymbol{\theta}, X, T)\frac{\mathrm{d}^m\theta}{(2\pi)^m} \equiv 0$$

since all $\boldsymbol{\kappa}^{(q)}(\boldsymbol{\theta}, X, T)$ are the right eigenvectors of $\hat{L}$ with zero eigenvalues.

It is easy to see also that all $\boldsymbol{\theta}_0(X, T)$ in the arguments of $\mathbf{\Phi}$ and $\boldsymbol{\kappa}^{(q)}$ will disappear after the integration so we get the statement of the lemma. □

Remark. As follows from the proof of lemma 1 we will always have in particular

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \kappa_{[\mathbf{U}(X,T)]i}^{(q)}(\boldsymbol{\theta}+\boldsymbol{\theta}_0(X, T))\Phi_{\theta^{\beta}}^{i}(\boldsymbol{\theta}+\boldsymbol{\theta}_0(X, T), \mathbf{U}(X, T))\frac{\mathrm{d}^m\theta}{(2\pi)^m} \equiv 0$$

for the case of the full regular family of quasiperiodic solutions (3.4).

The Whitham system can now be written in the form

$$\begin{aligned} &\frac{\partial k^{\alpha}}{\partial U^{\nu}}U_T^{\nu} = \frac{\partial\omega^{\alpha}}{\partial U^{\nu}}U_X^{\nu}, \qquad \alpha = 1, \ldots, m \\ &C_{\nu}^{(q)}(\mathbf{U})U_T^{\nu} = D_{\nu}^{(q)}(\mathbf{U})U_X^{\nu}, \qquad q = 1, \ldots, m+r \end{aligned} \tag{3.15}$$

where $\mathrm{rank}\|\partial k^{\alpha}/\partial U^{\nu}\| = m$ according to our assumption above. In the generic case, the derivatives U_T^{ν} can be expressed through U_X^{μ} and the Whitham system (3.15) can be written in the form (1.7).

Let us state that the method described above is not the only one to get the Whitham system for the system (3.1). In particular, the method of averaging of conservation laws [13, 15–22] also gives another way to get the system for the slow modulations of parameters $\mathbf{U}(X, T)$. It can be shown that both these methods give the equivalent systems (1.7) for the parameters $\mathbf{U}(X, T)$ (in regular situation). Thus the averaged conservation laws then give the additional conservation law for the system (3.15).

We will get here the symplectic representation of the conditions of compatibility of the system (3.11) which is also equivalent to (3.15) in the generic case. In general, we can state that the system (3.15) admits the averaged symplectic structure in the sense discussed above.

Let us now put some special conditions connected with 'invariant tori' corresponding to quasiperiodic solutions (3.4) which we will need for the averaging of the symplectic structure (2.3). Namely, we will require that we have m linearly independent local flows

$$\varphi^i_{t^\alpha} = Q^i_{(\alpha)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots) \tag{3.16}$$

(which can contain the system (3.1)) which commute with (3.1) and admit the same symplectic structure (2.3) with some local Hamiltonian functions $F_{(\alpha)}[\boldsymbol{\varphi}]$, i.e.

$$\int_{-\infty}^{+\infty} \Omega_{ij}(x, y) Q^i_{(\alpha)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_y, \ldots)\,\mathrm{d}y \equiv \frac{\delta}{\delta\varphi^i(x)} F_{(\alpha)}$$

where

$$F_{(\alpha)}[\boldsymbol{\varphi}] = \int_{-\infty}^{+\infty} f_{(\alpha)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots)\,\mathrm{d}x.$$

This automatically means that the functionals $H^{(s)}[\boldsymbol{\varphi}]$ should also give the conservation laws for the systems (3.16) and we can write

$$h^{(s)}_{t^\alpha} \equiv \partial_x J^{(s)}_\alpha(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots) \tag{3.17}$$

for some functions $J^{(s)}_\alpha(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots)$.

We will require that the flows (3.16) generate the 'linear shifts' of the angles θ^β_0 on the solutions (3.4) with some frequencies $\omega^\beta_{(\alpha)}(\mathbf{U})$ such that the matrix $\|\omega^\beta_{(\alpha)}\|$ is non-degenerate, i.e. we have

$$\omega^\beta_{(\alpha)}(\mathbf{U})\Phi^i_{\theta^\beta} = Q^i_{(\alpha)}(\boldsymbol{\Phi}, k^\delta(\mathbf{U})\boldsymbol{\Phi}_{\theta^\delta}, \ldots) \tag{3.18}$$

with $\det\|\omega^\beta_{(\alpha)}(\mathbf{U})\| \neq 0$.

Let us denote by $\|\gamma^\beta_\alpha\|$ the inverse matrix $\|\omega^\beta_{(\alpha)}\|^{-1}$ such that

$$\gamma^\delta_\alpha(\mathbf{U})\omega^\beta_{(\delta)}(\mathbf{U}) = \delta^\beta_\alpha. \tag{3.19}$$

We can also write

$$\Phi^i_{\theta^\alpha} = \gamma^\beta_\alpha(\mathbf{U}) Q^i_{(\beta)}(\boldsymbol{\Phi}, k^\delta(\mathbf{U})\boldsymbol{\Phi}_{\theta^\delta}, \ldots) \tag{3.20}$$

on the family (3.4).

4. The extended phase space and some technical lemmas

In this section, we will prove some technical lemmas concerning the form (2.3) on the 'extended' functional space. As we stated already, we consider the form (2.3) on the loop space $\mathcal{W}_0$ of functions $\varphi^i(x)$ rapidly decreasing or approaching some fixed constants C^i for $x \to \pm\infty$. Let us now define the extended space $\hat{\mathcal{W}}_0$ of smooth functions $\varphi^i(\boldsymbol{\theta}, x)$

($\boldsymbol{\theta} = (\theta^1, \ldots, \theta^m)$), 2π-periodic w.r.t. each θ^α and approaching the same constants C^i at each $\boldsymbol{\theta}$ for $x \to \pm\infty$. We define the 'extended' symplectic form $\tilde{\Omega}_{ij}(\theta, \theta', x, y)$ by the formula

$$\tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', x, y) = \sum_{k\geqslant 0} \omega_{ij}^{(k)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, x), \boldsymbol{\varphi}_x(\boldsymbol{\theta}, x), \ldots)\delta^{(k)}(x-y)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}') + \sum_{s=1}^{g} e_s \frac{\delta \tilde{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, x)} \nu(x-y)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\frac{\delta \tilde{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}', y)}, \qquad i, j = 1, \ldots, n \tag{4.1}$$

where the functionals $\tilde{H}^{(s)}$ are defined on $\hat{\mathcal{W}}_0$ by the formula[4]

$$\tilde{H}^{(s)}[\boldsymbol{\varphi}] = \int_{-\infty}^{+\infty}\int_0^{2\pi}\ldots\int_0^{2\pi} h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, x), \boldsymbol{\varphi}_x(\boldsymbol{\theta}, x), \ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}x.$$

Let us also note that we normalize $\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})$ such that

$$\int_0^{2\pi}\ldots\int_0^{2\pi}\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\frac{d^m\theta'}{(2\pi)^m} = 1.$$

It is easy to see that (4.1) gives the closed 2-form on $\hat{\mathcal{W}}_0$. Let us now prove the first technical lemma which we will need later.

Lemma 2. *For any $\alpha, \beta = 1, \ldots, m$ we have*

$$C_{\alpha\beta}[\boldsymbol{\varphi}] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{2\pi}\ldots\int_0^{2\pi}\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, x)\tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', x, y)\varphi^j_{\theta'^\beta}(\boldsymbol{\theta}', x)\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}^m\theta}{(2\pi)^m}\frac{\mathrm{d}^m\theta'}{(2\pi)^m} \equiv 0$$

on $\hat{\mathcal{W}}_0$.

Proof. Let us first prove the relation

$$\frac{\delta C_{\alpha\beta}[\boldsymbol{\varphi}]}{\delta\varphi^i(\boldsymbol{\theta}, x)} \equiv 0.$$

We will use the infinite-dimensional form of the relation

$$\frac{\partial}{\partial x^i}\langle\boldsymbol{\xi}\omega\boldsymbol{\eta}\rangle = [\mathcal{L}_{\boldsymbol{\xi}}\langle\omega\boldsymbol{\eta}\rangle]_i - [\mathcal{L}_{\boldsymbol{\eta}}\langle\omega\boldsymbol{\xi}\rangle]_i - \langle\omega[\boldsymbol{\xi}, \boldsymbol{\eta}]\rangle_i$$

which is valid for the closed form $\omega_{ij}(x)$ on a manifold and any vector fields $\xi^i(x)$ and $\eta^k(x)$. The notation $\langle\boldsymbol{\xi}\omega\boldsymbol{\eta}\rangle$, $\langle\omega\boldsymbol{\xi}\rangle$ and $\langle\omega\boldsymbol{\eta}\rangle$ means here the function $\xi^j\omega_{jk}\eta^k$ and the 1-forms $\omega_{jk}\xi^k$ and $\omega_{jk}\eta^k$ respectively. The operators $\mathcal{L}_{\boldsymbol{\xi}}$ and $\mathcal{L}_{\boldsymbol{\eta}}$ are the Lie derivatives w.r.t. vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and $[\boldsymbol{\xi}, \boldsymbol{\eta}]$ is the commutator of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

Indeed, we have for any closed $\omega_{ij}(x)$,

$$\begin{aligned}
\frac{\partial}{\partial x^i}(\xi^j\omega_{jk}\eta^k) &= \frac{\partial\xi^j}{\partial x^i}\omega_{jk}\eta^k + \xi^j\frac{\partial\omega_{jk}}{\partial x^i}\eta^k + \xi^j\omega_{jk}\frac{\partial\eta^k}{\partial x^i}\\
&= \frac{\partial\xi^j}{\partial x^i}\omega_{jk}\eta^k + \xi^j\omega_{jk}\frac{\partial\eta^k}{\partial x^i} - \xi^j\left(\frac{\partial\omega_{ki}}{\partial x^j} + \frac{\partial\omega_{ij}}{\partial x^k}\right)\eta^k\\
&= \frac{\partial\xi^j}{\partial x^i}\omega_{jk}\eta^k + \xi^j\frac{\partial}{\partial x^j}[\omega_{ik}\eta^k] - \xi^j\omega_{jk}\frac{\partial\eta^k}{\partial x^i} - \eta^k\frac{\partial}{\partial x^k}[\omega_{ij}\xi^j]\\
&\quad - \omega_{ik}\xi^j\frac{\partial\eta^k}{\partial x^j} + \omega_{ij}\eta^k\frac{\partial\xi^j}{\partial x^k}\\
&= [\mathcal{L}_{\boldsymbol{\xi}}\langle\omega\boldsymbol{\eta}\rangle]_i - [\mathcal{L}_{\boldsymbol{\eta}}\langle\omega\boldsymbol{\xi}\rangle]_i - \langle\omega[\boldsymbol{\xi}, \boldsymbol{\eta}]\rangle_i
\end{aligned}$$

(we assume summation over the repeated indices).

[4] We can always normalize the densities $h^{(s)}$ such that $h^{(s)}(\mathbf{C}, 0, \ldots) = 0$.

In our case $\partial/\partial x^i$ should be replaced by $\delta/\delta\varphi^i(\boldsymbol{\theta}, x)$ and we can define the vector fields

$$\xi^i(\boldsymbol{\theta}, x)[\varphi] = \varphi^i_{\theta^\alpha}, \qquad \eta^i(\boldsymbol{\theta}, x)[\varphi] = \varphi^i_{\theta^\beta}$$

and the corresponding dynamical systems on $\hat{\mathcal{W}}_0$

$$\varphi^i_{t_1} = \varphi^i_{\theta^\alpha}, \qquad \varphi^i_{t_2} = \varphi^i_{\theta^\beta}$$

(let us recall that x and $\boldsymbol{\theta}$ now also play the role of 'indices').

It is easy to see that the fields $\boldsymbol{\xi}[\varphi]$ and $\boldsymbol{\eta}[\varphi]$ commute with each other.

The expression $C_{\alpha\beta}[\varphi]$ can now be written as $\langle\boldsymbol{\xi}\tilde{\Omega}\boldsymbol{\eta}\rangle$ and we can write

$$\frac{\delta C_{\alpha\beta}[\varphi]}{\delta\varphi^i(\boldsymbol{\theta}, x)} = [\mathcal{L}_\xi\langle\tilde{\Omega}\boldsymbol{\eta}\rangle]_i(\boldsymbol{\theta}, x) - [\mathcal{L}_\eta\langle\tilde{\Omega}\boldsymbol{\xi}\rangle]_i(\boldsymbol{\theta}, x)$$

where

$$\begin{aligned}\langle\tilde{\Omega}\boldsymbol{\xi}\rangle_i(\boldsymbol{\theta}, x) &= \int_{-\infty}^{+\infty}\int_0^{2\pi}\dots\int_0^{2\pi}\tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', x, y)\varphi^j_{\theta^\alpha}(\boldsymbol{\theta}', y)\frac{\mathrm{d}^m\theta'}{(2\pi)^m}\,\mathrm{d}y\\ &= \int_{-\infty}^{+\infty}\Omega_{ij}(\boldsymbol{\theta}, x, y)\varphi^j_{\theta^\alpha}(\boldsymbol{\theta}, y)\,\mathrm{d}y.\end{aligned} \tag{4.2}$$

Also

$$\langle\tilde{\Omega}\boldsymbol{\eta}\rangle_i(\boldsymbol{\theta}, x) = \int_{-\infty}^{+\infty}\Omega_{ij}(\boldsymbol{\theta}, x, y)\varphi^j_{\theta^\beta}(\boldsymbol{\theta}, y)\,\mathrm{d}y \tag{4.3}$$

where $\varphi^i(\boldsymbol{\theta}, x)$, $\varphi^j(\boldsymbol{\theta}, y)$ are considered just as the functions of x and y at any fixed $\boldsymbol{\theta}$.

The operations of Lie derivatives $[\mathcal{L}_\xi\mathbf{q}]_i(\boldsymbol{\theta}, x)$ and $[\mathcal{L}_\eta\mathbf{q}]_i(\boldsymbol{\theta}, x)$ for any 1-form $q_i(\boldsymbol{\theta}, x)$ can be written as

$$\begin{aligned}[\mathcal{L}_\xi\mathbf{q}]_i(\boldsymbol{\theta}, x) = &\int_{-\infty}^{+\infty}\int_0^{2\pi}\dots\int_0^{2\pi}\varphi^k_{\theta'^\alpha}(\boldsymbol{\theta}', z)\frac{\delta}{\delta\varphi^k(\boldsymbol{\theta}', z)}q_i(\boldsymbol{\theta}, x)\frac{\mathrm{d}^m\theta'}{(2\pi)^m}\,\mathrm{d}z\\ &+\int_{-\infty}^{+\infty}\int_0^{2\pi}\dots\int_0^{2\pi}q_k(\boldsymbol{\theta}', z)\frac{\delta\varphi^k_{\theta'^\alpha}(\boldsymbol{\theta}', z)}{\delta\varphi^i(\boldsymbol{\theta}, x)}\frac{\mathrm{d}^m\theta'}{(2\pi)^m}\,\mathrm{d}z\end{aligned}$$

where

$$\frac{\delta\varphi^k_{\theta'^\alpha}(\boldsymbol{\theta}', z)}{\delta\varphi^i(\boldsymbol{\theta}, x)} = \delta^k_i\delta_{\theta'^\alpha}(\boldsymbol{\theta}' - \boldsymbol{\theta})\delta(z - x).$$

So we have

$$[\mathcal{L}_\xi\mathbf{q}]_i(\boldsymbol{\theta}, x) = -\frac{\partial}{\partial\theta^\alpha}q_i(\boldsymbol{\theta}, x) + \int_{-\infty}^{+\infty}\int_0^{2\pi}\dots\int_0^{2\pi}\varphi^k_{\theta'^\alpha}(\boldsymbol{\theta}', z)\frac{\delta q_i(\boldsymbol{\theta}, x)}{\delta\varphi^k(\boldsymbol{\theta}', z)}\frac{\mathrm{d}^m\theta'}{(2\pi)^m}\,\mathrm{d}z$$

which is zero if $q_i(\boldsymbol{\theta}, x)$ does not contain explicit dependence on $\boldsymbol{\theta}$. (The same for $[\mathcal{L}_\eta\mathbf{q}]_i(\boldsymbol{\theta}, x)$.)

Using expressions (4.2), (4.3) we see that neither of the forms $\langle\tilde{\Omega}\boldsymbol{\xi}\rangle_i(\boldsymbol{\theta}, x)$ and $\langle\tilde{\Omega}\boldsymbol{\eta}\rangle_i(\boldsymbol{\theta}, x)$ depend explicitly on $\boldsymbol{\theta}$ so we get $\delta C_{\alpha\beta}[\varphi]/\delta\varphi^i(\boldsymbol{\theta}, x) \equiv 0$ on $\hat{\mathcal{W}}_0$.

Using now the fact that $C_{\alpha\beta}[\varphi] \equiv 0$ on the functions $\varphi^i(\boldsymbol{\theta}, x)$ which are constants w.r.t. $\boldsymbol{\theta}$ at any given x we get the proof of the lemma. □

Let us introduce the nonlocal functionals

$$\begin{aligned}W^{(s)}(\boldsymbol{\theta}, x)[\varphi] = D^{-1}h^{(s)}(\varphi, \varphi_x, \dots) = &\frac{1}{2}\int_{-\infty}^{x}h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, y), \boldsymbol{\varphi}_y(\boldsymbol{\theta}, y), \dots)\,\mathrm{d}y\\ &-\frac{1}{2}\int_{x}^{+\infty}h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, y), \boldsymbol{\varphi}_y(\boldsymbol{\theta}, y), \dots)\,\mathrm{d}y.\end{aligned} \tag{4.4}$$

It is easy to see that for any $\varphi(\boldsymbol{\theta}, x)$ the functions $W^{(s)}(\boldsymbol{\theta}, x)$ are 2π-periodic w.r.t. each θ^α and we also have

$$W^{(s)}(\boldsymbol{\theta}, -\infty) = -W^{(s)}(\boldsymbol{\theta}, +\infty) \tag{4.5}$$

on $\hat{\mathcal{L}}_0$ at any fixed $\boldsymbol{\theta}$.

We will also need the following simple proposition:

Proposition 1. *The expressions*

$$h^{(s)}_{\theta^\alpha} - \frac{\delta \tilde{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, x)} \varphi^i_{\theta^\alpha}$$

can be written as total derivatives w.r.t. x of the local functions $T^{(s)}_\alpha(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots)$, i.e.

$$h^{(s)}_{\theta^\alpha} - \frac{\delta \tilde{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, x)} \varphi^i_{\theta^\alpha} \equiv \frac{\mathrm{d}}{\mathrm{d}x} T^{(s)}_\alpha(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots) \tag{4.6}$$

where

$$T^{(s)}_\alpha(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots) = \sum_{k\geqslant 1} \sum_{p=0}^{k-1} (-1)^p \left(\frac{\partial h^{(s)}}{\partial \varphi^i_{kx}}\right)_{px} \varphi^i_{\theta^\alpha, (k-p-1)x}. \tag{4.7}$$

Proof. Using the formulae

$$h^{(s)}_{\theta^\alpha} = \frac{\partial h^{(s)}}{\partial \varphi^i} \varphi^i_{\theta^\alpha} + \frac{\partial h^{(s)}}{\partial \varphi^i_x} \varphi^i_{\theta^\alpha, x} + \cdots$$

$$\frac{\delta \tilde{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, x)} = \frac{\partial h^{(s)}}{\partial \varphi^i} - \left(\frac{\partial h^{(s)}}{\partial \varphi^i_x}\right)_x + \cdots$$

we get the required statement just by direct calculation. □

Let us now prove another important lemma.

Lemma 3.

(1) *For any symplectic form (2.3) we have the relations*

$$\varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots) \varphi^j_{\theta^\beta, kx} + \sum_{s=1}^{g} e_s \left(h^{(s)}_{\theta^\beta} T^{(s)}_\alpha - h^{(s)}_{\theta^\alpha} T^{(s)}_\beta + \left(T^{(s)}_\alpha\right)_x T^{(s)}_\beta\right)$$
$$\equiv \frac{\partial}{\partial \theta^\gamma} Q^\gamma_{\alpha\beta}(\boldsymbol{\varphi}, \ldots) + \frac{\partial}{\partial x} A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots) \tag{4.8}$$

for some local functions $Q^\gamma_{\alpha\beta}(\boldsymbol{\varphi}, \ldots)$, $A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots)$ (summation over the repeated indices).

(2) *The functions $A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots)$ (defined modulo the constant functions) can be normalized in such a way that $A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots) \equiv 0$ for any $\varphi(\boldsymbol{\theta}, x)$ depending on x only (and constant with respect to $\boldsymbol{\theta}$ at every fixed x).*

Proof.

(1) Let us consider the values

$$F_{\alpha\beta}(\boldsymbol{\theta}, x) = \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, x) \int_{-\infty}^{+\infty} \int_0^{2\pi} \ldots \int_0^{2\pi} \tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', x, y) \varphi^j_{\theta'^\beta}(\boldsymbol{\theta}', y) \frac{\mathrm{d}^m\theta'}{(2\pi)^m} \, \mathrm{d}y.$$

We have according to lemma 2,

$$\int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} F_{\alpha\beta}(\boldsymbol{\theta}, x) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \,\mathrm{d}x \equiv 0.$$

We have on the other hand

$$F_{\alpha\beta}(\boldsymbol{\theta}, x) = \varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \varphi^j_{\theta^\beta, kx} + \sum_{s=1}^{g} e_s \left(\varphi^i_{\theta^\alpha} \frac{\delta \tilde{H}^{(s)}}{\delta \varphi^i(\boldsymbol{\theta}, x)} \int_{-\infty}^{+\infty} \nu(x-y) \frac{\delta \tilde{H}^{(s)}}{\delta \varphi^j(\boldsymbol{\theta}, y)} \varphi^j_{\theta^\beta} \,\mathrm{d}y \right).$$

According to proposition 1 we can write

$$\begin{aligned} F_{\alpha\beta}(\boldsymbol{\theta}, x) &= \varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \varphi^j_{\theta^\beta, kx} \\ &\quad + \sum_{s=1}^{g} e_s \big(h^{(s)}_{\theta^\alpha} - \big(T^{(s)}_\alpha\big)_x \big) \int_{-\infty}^{+\infty} \nu(x-y) \big(h^{(s)}_{\theta^\beta} - \big(T^{(s)}_\beta\big)_y \big) \,\mathrm{d}y \\ &= \varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij} \varphi^j_{\theta^\beta, kx} + \sum_{s=1}^{g} e_s \big(W^{(s)}_{\theta^\alpha x} W^{(s)}_{\theta^\beta} \\ &\quad - W^{(s)}_{\theta^\alpha x} T^{(s)}_\beta - W^{(s)}_{\theta^\beta} \big(T^{(s)}_\alpha\big)_x + \big(T^{(s)}_\alpha\big)_x T^{(s)}_\beta \big) \end{aligned}$$

(we used here relations (4.5) at infinity).

We can now rewrite $F_{\alpha\beta}(\boldsymbol{\theta}, x)$ in the following form:

$$\begin{aligned} F_{\alpha\beta}(\boldsymbol{\theta}, x) &= \varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij} \varphi^j_{\theta^\beta, kx} + \sum_{s=1}^{g} e_s \big[W^{(s)}_{\theta^\beta x} T^{(s)}_\alpha - W^{(s)}_{\theta^\alpha x} T^{(s)}_\beta + \big(T^{(s)}_\alpha\big)_x T^{(s)}_\beta \big] + \sum_{s=1}^{g} e_s \\ &\quad \times \left[\frac{1}{2} \big(W^{(s)}_x W^{(s)}_{\theta^\beta} \big)_{\theta^\alpha} - \frac{1}{2} \big(W^{(s)}_x W^{(s)}_{\theta^\alpha} \big)_{\theta^\beta} + \frac{1}{2} \big(W^{(s)}_{\theta^\alpha} W^{(s)}_{\theta^\beta} \big)_x - \big(W^{(s)}_{\theta^\beta} T^{(s)}_\alpha \big)_x \right]. \end{aligned}$$

It is easy to see that

$$\sum_{s=1}^{g} e_s \int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{1}{2} \big[\big(W^{(s)}_x W^{(s)}_{\theta^\beta} \big)_{\theta^\alpha} - \big(W^{(s)}_x W^{(s)}_{\theta^\alpha} \big)_{\theta^\beta} \big] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \,\mathrm{d}x \equiv 0$$

in view of the periodicity of $W^{(s)}(\boldsymbol{\theta}, x)$ w.r.t. all θ^α and

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} \left[\frac{1}{2} \big(W^{(s)}_{\theta^\alpha} W^{(s)}_{\theta^\beta} \big)_x - \big(W^{(s)}_{\theta^\beta} T^{(s)}_\alpha \big)_x \right] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \,\mathrm{d}x \\ &\qquad = \int_0^{2\pi} \dots \int_0^{2\pi} \left[\frac{1}{2} W^{(s)}_{\theta^\alpha} W^{(s)}_{\theta^\beta} \Big|^{x=+\infty}_{x=-\infty} - W^{(s)}_{\theta^\beta} T^{(s)}_\alpha \Big|^{x=+\infty}_{x=-\infty} \right] \frac{\mathrm{d}^m\theta}{(2\pi)^m}. \end{aligned} \tag{4.9}$$

Both terms in (4.9) are zero in view of (4.5) and $T^{(s)}_\alpha \to 0$ for $x \to \pm\infty$ on $\hat{\mathcal{W}}_0$.

Using now the relations $W^{(s)}_{\theta^\alpha x} = h^{(s)}_{\theta^\alpha}$, $W^{(s)}_{\theta^\beta x} = h^{(s)}_{\theta^\beta}$ we have

$$\int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} \left[\varphi^i_{\theta^\alpha} \sum_{k\geqslant 0} \omega^{(k)}_{ij} \varphi^j_{\theta^\beta, kx} + \sum_{s=1}^{g} e_s \big(h^{(s)}_{\theta^\beta} T^{(s)}_\alpha - h^{(s)}_{\theta^\alpha} T^{(s)}_\beta + \big(T^{(s)}_\alpha\big)_x T^{(s)}_\beta \big) \right] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \,\mathrm{d}x \equiv 0$$

so we get (4.8).

(2) We can now normalize the functions $A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots)$ such that $A_{\alpha\beta} = 0$ for $\varphi^i(\boldsymbol{\theta}, x) \equiv \text{const} = C^i$. Now for any function $\varphi^i(\boldsymbol{\theta}, x)$ depending only on x we have $(\partial/\partial x) A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots) = 0$ according to relation (4.8). Using the fact that $A_{\alpha\beta}(\boldsymbol{\theta}, \pm\infty) = 0$ on $\hat{\mathcal{W}}_0$ we get part (2) of the lemma on $\hat{\mathcal{W}}_0$. Now using the fact that $A_{\alpha\beta}(\boldsymbol{\varphi}, \ldots)$ is a local expression of $\boldsymbol{\varphi}(\boldsymbol{\theta}, x)$ and its derivatives we get in fact that $A_{\alpha\beta}(\boldsymbol{\theta}, \pm\infty) \equiv 0$ for any $\varphi(\boldsymbol{\theta}, x)$ depending on x only for this normalization of $A_{\alpha\beta}$ which actually does not depend on the constants C^i. □

We will also need the following technical lemma.

Lemma 4. *For any symplectic form (2.3) we have*

$$\begin{aligned}
-\int_0^{2\pi} \cdots \int_0^{2\pi} & A_{\alpha\beta}(\boldsymbol{\varphi}(\boldsymbol{\theta}, y), \boldsymbol{\varphi}_y(\boldsymbol{\theta}, y), \ldots) \frac{\mathrm{d}^m\theta}{(2\pi)^m} + \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{k\geqslant 1} \sum_{p=1}^{k} C_k^p (-1)^{p-1} \\
& \times \left(\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, y) \omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta}, y), \ldots) \varphi^j_{\theta^\beta, (k-p)y}\right)_{(p-1)y} \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
& - \frac{1}{2} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{s=1}^{g} e_s \big[W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) \\
& - W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, +\infty) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, +\infty)\big] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
& + \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{s=1}^{g} e_s T^{(s)}_\alpha(\boldsymbol{\theta}, y) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
& + \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{s=1}^{g} e_s \left(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) - T^{(s)}_\alpha(\boldsymbol{\theta}, y)\right) \\
& \times \left(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) - T^{(s)}_\beta(\boldsymbol{\theta}, y)\right) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
\equiv & \int_0^{2\pi} \cdots \int_0^{2\pi} A_{\beta\alpha}(\boldsymbol{\varphi}(\boldsymbol{\theta}, y), \boldsymbol{\varphi}_y(\boldsymbol{\theta}, y), \ldots) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
& + \frac{1}{2} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{s=1}^{g} e_s \big[W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) \\
& - W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, +\infty) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, +\infty)\big] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
& - \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{s=1}^{g} e_s W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) T^{(s)}_\beta(\boldsymbol{\theta}, y) \frac{\mathrm{d}^m\theta}{(2\pi)^m}
\end{aligned} \tag{4.10}$$

where the values $A_{\alpha\beta}$ (normalized in the 'right' way), $W^{(s)}$ and $T^{(s)}_\alpha$ are introduced in (4.8), (4.4) and (4.7) respectively.

Proof. Let us consider the quantities

$$\begin{aligned}
E_{\alpha\beta}(y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} & \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, z) \tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', z, w) \varphi^j_{\theta'^\beta}(\boldsymbol{\theta}', w) \\
& \times \nu(w - y)\, \mathrm{d}z\, \mathrm{d}w \frac{\mathrm{d}^m\theta}{(2\pi)^m} \frac{\mathrm{d}^m\theta'}{(2\pi)^m}.
\end{aligned}$$

We have

$$E_{\alpha\beta}(y) = \int \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, z) \sum_{k\geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_w, \ldots)\delta^{(k)}(z-w)\varphi^j_{\theta^\beta}(\boldsymbol{\theta}, w)\nu(w-y)\,\mathrm{d}z\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$+ \int \sum_{s=1}^{g} e_s\big(W^{(s)}_{\theta^\alpha z} - \big(T^{(s)}_\alpha\big)_z\big)\nu(z-w)$$
$$\times \big(W^{(s)}_{\theta^\beta w} - \big(T^{(s)}_\beta\big)_w\big)\nu(w-y)\,\mathrm{d}z\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}.$$

We can calculate these values in two ways:

(I) Let us first make the integration with respect to z. We have

$$E_{\alpha\beta}(y) = \int \sum_{k\geqslant 0}(-1)^k\big(\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, w)\omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_w, \ldots)\big)_{kw}\varphi^j_{\theta^\beta}(\boldsymbol{\theta}, w)\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$- \int \sum_{s=1}^{g} e_s\big(W^{(s)}_{\theta^\alpha} - T^{(s)}_\alpha(z)\big)\big(W^{(s)}_{\theta^\beta w} - \big(T^{(s)}_\beta\big)_w\big)\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$= \int \Bigg[\sum_{k\geqslant 0}(-1)^k\big(\varphi^i_{\theta^\alpha}\omega^{(k)}_{ij}\big)_{kw}\varphi^j_{\theta^\beta} - \sum_{s=1}^{g} e_s\big(h^{(s)}_{\theta^\alpha}T^{(s)}_\beta$$
$$- h^{(s)}_{\theta^\beta}T^{(s)}_\alpha + \big(T^{(s)}_\beta\big)_w T^{(s)}_\alpha\big)\Bigg]\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$- \int \sum_{s=1}^{g} e_s\left[\frac{1}{2}\big(W^{(s)}_{\theta^\alpha}W^{(s)}_{\theta^\beta}\big)_w - \frac{1}{2}\big(W^{(s)}_{\theta^\beta}W^{(s)}_w\big)_{\theta^\alpha}\right.$$
$$\left.+\frac{1}{2}\big(W^{(s)}_{\theta^\alpha}W^{(s)}_w\big)_{\theta^\beta} - \big(W^{(s)}_{\theta^\alpha}T^{(s)}_\beta\big)_w\right]\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}.$$

Using now the skew-symmetry of the form $\tilde{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', z, w)$ and relations (4.8) we can write

$$E_{\alpha\beta}(y) = -\int (A_{\beta\alpha}(\boldsymbol{\theta}, w))_w\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$- \int \sum_{s=1}^{g} e_s\left[\frac{1}{2}\big(W^{(s)}_{\theta^\alpha}W^{(s)}_{\theta^\beta}\big)_w - \big(W^{(s)}_{\theta^\alpha}T^{(s)}_\beta\big)_w\right]\nu(w-y)\,\mathrm{d}w\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$= \int A_{\beta\alpha}(\boldsymbol{\theta}, y)\frac{\mathrm{d}^m\theta}{(2\pi)^m} - \int W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y)T^{(s)}_\beta(\boldsymbol{\theta}, y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$+\frac{1}{2}\int \sum_{s=1}^{g} e_s W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$-\frac{1}{4}\int \big[W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, +\infty)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, +\infty) + W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, -\infty)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, -\infty)\big]\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

(we used the relation $A_{\beta\alpha}(\boldsymbol{\theta}, \pm\infty) = 0$ on $\hat{\mathcal{W}}_0$).

(II) Let us now first make the integration with respect to w. We have

$$E_{\alpha\beta}(y) = \int \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, z)\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_z, \ldots)\varphi^j_{\theta^\beta, kz}(\boldsymbol{\theta}, z)\nu(z-y)\,\mathrm{d}z\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

$$
\begin{aligned}
&+ \int \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, z) \sum_{k\geqslant 1} \sum_{p=1}^{k} C_k^p \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_z, \ldots) \\
&\times \varphi^j_{\theta^\beta,(k-p)z}(\boldsymbol{\theta}, z)\delta^{(p-1)}(z-y)\,\mathrm{d}z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&+ \int \sum_{s=1}^{g} e_s\big[W^{(s)}_{\theta^\alpha z}(\boldsymbol{\theta}, z) - \big(T^{(s)}_\alpha\big)_z(\boldsymbol{\theta}, z)\big] \\
&\times \bigg[W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, z) - T^{(s)}_\beta(\boldsymbol{\theta}, z) + \frac{1}{4}W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, -\infty) \\
&- \frac{1}{4}W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, +\infty)\bigg]\nu(z-y)\,\mathrm{d}z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&- \int \sum_{s=1}^{g} e_s\big[W^{(s)}_{\theta^\alpha z}(\boldsymbol{\theta}, z) - \big(T^{(s)}_\alpha(\boldsymbol{\theta}, z)\big)_z\big] \\
&\times \nu(z-y)\big[W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) - T^{(s)}_\beta(\boldsymbol{\theta}, y)\big]\,\mathrm{d}z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
= &- \int A_{\alpha\beta}(\boldsymbol{\theta}, y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} + \int \sum_{k\geqslant 1} \sum_{p=1}^{k} C_k^p (-1)^{p-1} \\
&\times \big(\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, y)\omega^{(k)}_{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_y, \ldots)\varphi^j_{\theta^\beta,(k-p)y}(\boldsymbol{\theta}, y)\big)_{(p-1)y} \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&- \frac{1}{2}\int \sum_{s=1}^{g} e_s W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&+ \frac{1}{4}\int \sum_{s=1}^{g} e_s\big[W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, -\infty) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, -\infty) \\
&+ W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, +\infty) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, +\infty)\big] \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&+ \int \sum_{s=1}^{g} e_s T^{(s)}_\alpha(\boldsymbol{\theta}, y) W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\
&+ \int \sum_{s=1}^{g} e_s\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, y) - T^{(s)}_\alpha(\boldsymbol{\theta}, y)\big)\big(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, y) - T^{(s)}_\beta(\boldsymbol{\theta}, y)\big) \frac{\mathrm{d}^m\theta}{(2\pi)^m}.
\end{aligned}
$$

Comparing (I) and (II) and using (4.5) we now get the statement of the lemma. □

5. The averaging of the weakly nonlocal symplectic structures

Let us now make the change $X = \epsilon x$, $T = \epsilon t$. We can again define a symplectic form in new coordinates which can be written as

$$
\begin{aligned}
\hat{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, Y) = &\sum_{k\geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta}, X), \epsilon\boldsymbol{\varphi}_X(\boldsymbol{\theta}, X), \ldots)\epsilon^k \delta^{(k)}(X-Y)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}') \\
&+ \frac{1}{\epsilon}\sum_{s=1}^{g} e_s \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, X)}\nu(X-Y)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}', Y)}, \qquad i, j = 1, \ldots, n
\end{aligned}
\tag{5.1}
$$

where

$$\hat{H}^{(s)} = \int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} h^s(\varphi(\boldsymbol{\theta}, X), \epsilon\varphi_X(\boldsymbol{\theta}, X), \dots) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \,\mathrm{d}X.$$

We will assume for simplicity that the family Λ of solutions of (3.5) contains the solutions corresponding to $k^\alpha = 0$ for some parameters $\mathbf{U} = \mathbf{U}_0$ such that $\Phi^i(\boldsymbol{\theta}, \mathbf{U}_0) = C^i = \text{const}$ (we should have $Q^i(\mathbf{C}, 0, \dots) = 0$ in this case).

Let us introduce the functional 'submanifold' $\mathcal{M}_0$ in the space of functions $\varphi(\boldsymbol{\theta}, X)$ (2π-periodic w.r.t. each θ^α) in the following way:

(1) We require that the functions $\varphi(\boldsymbol{\theta}, X)$ from $\mathcal{M}_0$ belong to the family Λ of solutions of (3.5) at any fixed X.
(2) We put $\mathbf{U}(X) \to \mathbf{U}_0$ (i.e. $\varphi^i(\boldsymbol{\theta}, X) \to C^i$) for $X \to \pm\infty$ (rapidly enough).

The functions $\mathbf{U}(X)$, $\boldsymbol{\theta}_0(X)$ can be taken as the coordinates on the submanifold $\mathcal{M}_0$ such that we have

$$\varphi^i_{[\mathbf{U},\boldsymbol{\theta}_0]}(\boldsymbol{\theta}, X) = \Phi^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X), \mathbf{U}(X))$$

for the functions belonging to $\mathcal{M}_0$.

We will also consider the 'ϵ-deformations' $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$ of the submanifold $\mathcal{M}_0$ defined with the aid of an arbitrary function $\boldsymbol{\Psi}_{(1)}(\boldsymbol{\theta}, X)$ 2π-periodic w.r.t. each θ^α and such that

$$\Psi^i_{(1)}(\boldsymbol{\theta}, X) \to 0 \qquad \text{for} \quad X \to \pm\infty.$$

Namely, we put

$$\varphi^i_{[\mathbf{U},\boldsymbol{\theta}_0]}(\boldsymbol{\theta}, X) = \Phi^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X), \mathbf{U}(X)) + \epsilon\Psi^i_{(1)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X), X)$$

which defines the ϵ-deformation of the function $\varphi_{[\mathbf{U},\boldsymbol{\theta}_0]}$ corresponding to the coordinates $\mathbf{U}(X)$, $\boldsymbol{\theta}_0(X)$. It is easy to see that the case $\boldsymbol{\Psi}_{(1)} = 0$ corresponds to the submanifold $\mathcal{M}_0$.

Let us now introduce the new coordinates $\boldsymbol{\theta}^*_0(X)$ on $\mathcal{M}_0$ and $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$ in the following way,

$$\theta_0^{*\alpha}(X) = \theta_0^\alpha(X) - \frac{1}{\epsilon} S^\alpha(X)$$

where

$$S^\alpha(X) = \int_{-\infty}^{+\infty} \nu(X - Y) k^\alpha(\mathbf{U}(Y)) \,\mathrm{d}Y.$$

We can then write on any $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$

$$\varphi^i_{[\mathbf{U},\boldsymbol{\theta}^*_0]}(\boldsymbol{\theta}, X) = \Phi^i\left(\boldsymbol{\theta} + \boldsymbol{\theta}^*_0(X) + \frac{1}{\epsilon}\mathbf{S}(X), \mathbf{U}(X)\right) + \epsilon\Psi^i_{(1)}\left(\boldsymbol{\theta} + \boldsymbol{\theta}^*_0(X) + \frac{1}{\epsilon}\mathbf{S}(X), X\right). \qquad (5.2)$$

We can see that the functions $\varphi^i_{[\mathbf{U},\boldsymbol{\theta}^*_0]}(\boldsymbol{\theta}, X)$ become rapidly oscillating functions of X (for fixed $\boldsymbol{\theta}$) for any fixed 'coordinates' $\mathbf{U}(X)$, $\boldsymbol{\theta}^*_0(X)$ and $\epsilon \to 0$. It is easy to see also that (5.2) represents in fact the first two terms of the expansion of asymptotic solutions (3.8) for the appropriate $\boldsymbol{\Psi}_{(1)}$.

Let us now formulate the theorem about the restriction of 2-form $\hat{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, Y)$ on the submanifolds $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$.

Theorem 4. *The restriction of the form $\hat{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, Y)$ to any submanifold $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$ in coordinates $U^\nu(X)$, $\theta_0^{*\alpha}(X)$ can be written as*

$$\Omega^{\text{rest}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega^1_{\nu\mu}(X,Y)\delta U^\nu(X)\delta U^\mu(Y)\,\mathrm{d}X\,\mathrm{d}Y$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega^2_{\nu\alpha}(X,Y)\delta U^\nu(X)\delta\theta_0^{*\alpha}(Y)\,\mathrm{d}X\,\mathrm{d}Y$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega^3_{\alpha\nu}(X,Y)\delta\theta_0^{*\alpha}(X)\delta U^\nu(Y)\,\mathrm{d}X\,\mathrm{d}Y$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega^4_{\alpha\beta}(X,Y)\delta\theta_0^{*\alpha}(X)\delta\theta_0^{*\beta}(Y)\,\mathrm{d}X\,\mathrm{d}Y$$

where

(I) the weak limit[5] $\Omega^{1(wl)}_{\nu\mu}(X,Y)$ *of the form* $\Omega^1_{\nu\mu}(X,Y)$ *can be written as*

$$\Omega^{1(wl)}_{\nu\mu}(X,Y) = \frac{1}{\epsilon}\sum_{\alpha=1}^{m}\left(\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial I_\alpha}{\partial U^\mu}(Y) + \frac{\partial I_\alpha}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial k^\alpha}{\partial U^\mu}(Y)\right)$$
$$+ \frac{1}{\epsilon}\sum_{s=1}^{g} e_s \frac{\partial\langle h^{(s)}\rangle}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial\langle h^{(s)}\rangle}{\partial U^\mu}(Y) + \frac{o(1)}{\epsilon}$$

where expressions $\langle h^{(s)}\rangle(\mathbf{U})$ are the averaged densities $h^{(s)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \ldots)$ and the functions $I^\alpha(\mathbf{U})$ are defined through the formulae

$$\frac{\partial I_\alpha}{\partial U^\nu} = -\frac{\partial k^\beta}{\partial U^\nu}\langle A_{\alpha\beta}\rangle + \frac{\partial k^\beta}{\partial U^\nu}\sum_{s=1}^{g} e_s\left[\gamma_\alpha^\delta\left(\left\langle T_\beta^{(s)}J_\delta^{(s)}\right\rangle - \left\langle T_\beta^{(s)}\right\rangle\left\langle J_\delta^{(s)}\right\rangle\right)\right.$$
$$\left. - \frac{1}{2}\gamma_\alpha^\delta\gamma_\beta^\zeta\left(\left\langle J_\delta^{(s)}J_\zeta^{(s)}\right\rangle - \left\langle J_\delta^{(s)}\right\rangle\left\langle J_\zeta^{(s)}\right\rangle\right)\right]$$
$$+ \left\langle \Phi^i_{U^\nu}\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\varphi,\ldots)\varphi^j_{\theta^\alpha,kx}\right\rangle - \sum_{s=1}^{g} e_s\left\langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)}T_\alpha^{(s)}\right\rangle$$
$$+ \sum_{s=1}^{g} e_s\gamma_\alpha^\beta\left(\left\langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)}J_\beta^{(s)}\right\rangle - \left\langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)}\right\rangle\left\langle J_\beta^{(s)}\right\rangle\right) \qquad (5.3)$$

(the functions $A_{\alpha\beta}$ are normalized according to lemma 3);

(II) the forms $\Omega^2_{\nu\alpha}(X,Y)$, $\Omega^3_{\alpha\nu}(X,Y)$ have the order $O(1)$ for $\epsilon \to 0$ on $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$;

(III) the form $\Omega^4_{\alpha\beta}(X,Y)$ has the order $O(\epsilon)$ for $\epsilon \to 0$ on $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$.

Proof. (I) Let us first rewrite relations (4.8) and (4.10) in the variables $\boldsymbol{\theta}$, X, i.e.

$$\varphi^i_{\theta^\alpha}\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}, \epsilon\boldsymbol{\varphi}_X, \ldots)\epsilon^k\varphi^j_{\theta^\beta,kX} + \sum_{s=1}^{g} e_s\left(h^{(s)}_{\theta^\beta}T_\alpha^{(s)} - h^{(s)}_{\theta^\alpha}T_\beta^{(s)} + \epsilon\left(T_\alpha^{(s)}\right)_X T_\beta^{(s)}\right)$$
$$\equiv \frac{\partial}{\partial\theta^\gamma}Q^\gamma_{\alpha\beta}(\boldsymbol{\varphi}, \epsilon\boldsymbol{\varphi}_X, \ldots) + \epsilon\frac{\partial}{\partial X}A_{\alpha\beta}(\boldsymbol{\varphi}, \epsilon\boldsymbol{\varphi}_X, \ldots) \qquad (5.4)$$

[5] We mean here the limit in the sense of functionals $\int \Omega^1_{\nu\mu}(X,Y)\xi^\nu(X)\eta^\mu(Y)\,\mathrm{d}X\,\mathrm{d}Y$ for any fixed smooth $\xi^\nu(X)$, $\eta^\mu(Y)$.

and

$$
\begin{aligned}
&-\int_0^{2\pi}\dots\int_0^{2\pi} A_{\alpha\beta}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\epsilon\boldsymbol{\varphi}_Y(\boldsymbol{\theta},Y),\dots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad+\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{k\geqslant 1}\sum_{p=1}^{k} C_k^p(-1)^{p-1}\epsilon^{k-1}\big(\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},Y)\\
&\qquad\times\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\dots)\varphi^j_{\theta^\beta,(k-p)Y}(\boldsymbol{\theta},Y)\big)_{(p-1)Y}\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad-\frac{1}{2}\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{s=1}^{g} e_s\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)\\
&\qquad-W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},+\infty)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},+\infty)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad+\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{s=1}^{g} e_s T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad+\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{s=1}^{g} e_s\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)-T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\\
&\qquad\times\big(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)-T^{(s)}_{\beta}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\quad\equiv\int_0^{2\pi}\dots\int_0^{2\pi} A_{\beta\alpha}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\epsilon\boldsymbol{\varphi}_Y(\boldsymbol{\theta},Y),\dots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad+\frac{1}{2}\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{s=1}^{g} e_s\big[W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)\\
&\qquad-W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},+\infty)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},+\infty)\big]\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&\qquad-\int_0^{2\pi}\dots\int_0^{2\pi}\sum_{s=1}^{g} e_s W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)T^{(s)}_{\beta}(\boldsymbol{\theta},Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}.
\end{aligned}\tag{5.5}
$$

We can write on $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$

$$
\begin{aligned}
\frac{\delta\varphi^i(\boldsymbol{\theta},X)}{\delta U^\nu(Y)}&=\frac{1}{\epsilon}\Phi^i_{\theta^\alpha}\left(\boldsymbol{\theta}+\boldsymbol{\theta}^*_0(X)+\frac{1}{\epsilon}\mathbf{S}(X),\mathbf{U}(X)\right)\nu(X-Y)\frac{\partial k^\alpha}{\partial U^\nu}(Y)\\
&\quad+\Psi^i_{(1)\theta^\alpha}\left(\boldsymbol{\theta}+\boldsymbol{\theta}^*_0(X)+\frac{1}{\epsilon}\mathbf{S}(X),X\right)\nu(X-Y)\frac{\partial k^\alpha}{\partial U^\nu}(Y)\\
&\quad+\Phi^i_{U^\nu}\left(\boldsymbol{\theta}+\boldsymbol{\theta}^*_0(X)+\frac{1}{\epsilon}\mathbf{S}(X),\mathbf{U}(X)\right)\delta(X-Y)\\
&=\frac{1}{\epsilon}\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},X)\nu(X-Y)\frac{\partial k^\alpha}{\partial U^\nu}(Y)+\Phi^i_{U^\nu}\left(\boldsymbol{\theta}+\boldsymbol{\theta}^*_0(X)+\frac{1}{\epsilon}\mathbf{S}(X),\mathbf{U}(X)\right)\delta(X-Y)
\end{aligned}
$$

and

$$
\frac{\delta\varphi^i(\boldsymbol{\theta},X)}{\delta\theta^{*\alpha}_0(Y)}=\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},X)\delta(X-Y).
$$

We can then write

$$
\begin{aligned}
\Omega^{1}_{\nu\mu}(X,Y) = &-\int \frac{1}{\epsilon^{2}} \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\nu(X-Z)\sum_{k\geqslant 0}\varphi^{i}_{\theta^{\alpha}}(\boldsymbol{\theta},Z)\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Z),\ldots)\\
&\times \epsilon^{k}\varphi^{j}_{\theta^{\beta},kZ}(\boldsymbol{\theta},Z)\nu(Z-Y)\frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\,\mathrm{d}Z\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&-\frac{1}{\epsilon^{2}}\int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\Bigg[\nu(X-Y)\sum_{k\geqslant 1}\sum_{p=1}^{k}C^{p}_{k}(-1)^{p-1}\epsilon^{k}\varphi^{i}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\\
&\times \omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\varphi^{j}_{\theta^{\beta},(k-p)Y}(\boldsymbol{\theta},Y)\Bigg]_{(p-1)Y}\frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&+\frac{1}{\epsilon}\int \Phi^{i}_{U^{\nu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\sum_{k\geqslant 0}\epsilon^{k}\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\ldots)\\
&\times\left[\varphi^{j}_{\theta^{\beta}}(\boldsymbol{\theta},X)\nu(X-Y)\right]_{kX}\frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&-\frac{1}{\epsilon}\int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\sum_{k\geqslant 0}(-1)^{k}\epsilon^{k}\left[\nu(X-Y)\varphi^{i}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\right]_{kY}\\
&\times \Phi^{j}_{U^{\mu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Y))\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&+\int\sum_{k\geqslant 0}\epsilon^{k}\Phi^{i}_{U^{\nu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\ldots)\\
&\times\left[\Phi^{j}_{U^{\mu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\delta(X-Y)\right]_{kX}\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&-\frac{1}{\epsilon^{3}}\int\sum_{s=1}^{g}e_{s}\frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\nu(X-Z)\\
&\times\left[\epsilon W^{(s)}_{\theta^{\alpha}Z}(\boldsymbol{\theta},Z)-\epsilon T^{(s)}_{\alpha,Z}(\boldsymbol{\theta},Z)\right]\nu(Z-W)\left[\epsilon W^{(s)}_{\theta^{\beta}W}(\boldsymbol{\theta},W)-\epsilon T^{(s)}_{\beta,W}(\boldsymbol{\theta},W)\right]\\
&\times\nu(W-Y)\frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\,\mathrm{d}Z\,\mathrm{d}W\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&+\frac{1}{\epsilon^{2}}\int\sum_{s=1}^{g}e_{s}\Phi^{i}_{U^{\nu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^{i}(\boldsymbol{\theta},X)}\nu(X-W)\\
&\times\left[\epsilon W^{(s)}_{\theta^{\beta}W}(\boldsymbol{\theta},W)-\epsilon T^{(s)}_{\beta,W}(\boldsymbol{\theta},W)\right]\nu(W-Y)\frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\mathrm{d}W\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}\\
&-\frac{1}{\epsilon^{2}}\int\sum_{s=1}^{g}e_{s}\frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\nu(X-Z)\left[\epsilon W^{(s)}_{\theta^{\alpha}Z}(\boldsymbol{\theta},Z)-\epsilon T^{(s)}_{\alpha,Z}(\boldsymbol{\theta},Z)\right]\nu(Z-Y)\\
&\times\frac{\delta\hat{H}^{(s)}}{\delta\varphi^{j}(\boldsymbol{\theta},Y)}\Phi^{j}_{U^{\mu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Y))\,\mathrm{d}Z\frac{\mathrm{d}^{m}\theta}{(2\pi)^{m}}
\end{aligned}
$$

$$+\frac{1}{\epsilon}\int\sum_{s=1}^{g}e_s\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}\nu(X-Y)$$

$$\times\frac{\delta\hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta},Y)}\Phi^j_{U^\mu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Y))\frac{\mathrm{d}^m\theta}{(2\pi)^m}.$$

We should now substitute the functions φ^i in the form (5.2) and we are interested here in the terms of the ϵ-expansion of $\Omega^1_{\nu\mu}(X,Y)$ containing $1/\epsilon$ and omit all the terms of order $O(1)$ for $\epsilon\to 0$. We can see then that we can omit the differentiation of the function $\nu(X-Y)$ in the second, third and fourth terms of the expression for $\Omega^1_{\nu\mu}(X,Y)$ since they appear only in regular terms for $\epsilon\to 0$. By the same reason we can omit the whole fifth term in the same expression which is regular for $\epsilon\to 0$.

Let us now consider especially the functions $W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},X)$. We first consider the submanifold $\mathcal{M}_0$ and represent the functions $\varphi_{[\mathbf{U},\boldsymbol{\theta}^*]}$ in the form

$$\varphi^i(\boldsymbol{\theta},X)=\Phi^i\left(\boldsymbol{\theta}+\boldsymbol{\theta}^*_0(X)+\frac{1}{\epsilon}\mathbf{S}(X),\mathbf{U}(X)\right). \tag{5.6}$$

Let us recall the commuting flows (3.16) for the system (3.1) and the corresponding relations (3.17) for the functions $h^{(s)}(\varphi,\varphi_x,\ldots)$. We can write in the new 'slow' variables X,T,

$$\epsilon\varphi^i_{T^\alpha}=Q^i_{(\alpha)}(\varphi,\epsilon\varphi_X,\ldots)$$

and relations (3.17) now become

$$h^{(s)}_{T^\alpha}\equiv\partial_X J^{(s)}_\alpha(\varphi,\epsilon\varphi_X,\ldots).$$

Let us represent the operator $\epsilon\partial_X$ on the functions (5.6) in the following form,

$$\epsilon\partial_X=\partial^I_X+\epsilon\partial^{II}_X$$

where

$$\partial^I_X=S^\alpha_X\frac{\partial}{\partial\theta^\alpha}=k^\alpha(X)\frac{\partial}{\partial\theta^\alpha},\qquad \partial^{II}_X=U^\nu_X\frac{\partial}{\partial U^\nu}+\theta^{*\alpha}_{0X}\frac{\partial}{\partial\theta^\alpha}.$$

We can write on the manifold $\mathcal{M}_0$

$$\omega^\eta_{(\alpha)}(\mathbf{U}(X))\frac{\partial}{\partial\theta^\eta}h^{(s)}\big(\varphi,\partial^I_X\varphi,\ldots\big)=\partial^I_X J^{(s)}_\alpha\big(\varphi,\partial^I_X\varphi,\ldots\big)$$

or using relations (3.19),

$$\begin{aligned}\frac{\partial}{\partial\theta^\alpha}h^{(s)}(\varphi,\partial^I_X\varphi,\ldots)&=\gamma^\delta_\alpha(\mathbf{U}(X))\partial^I_X J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)\\&=\gamma^\delta_\alpha(\mathbf{U}(X))\big[\epsilon\partial_X J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)-\epsilon\partial^{II}_X J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)\big].\end{aligned}$$

We have then on $\mathcal{M}_0$

$$\begin{aligned}h^{(s)}_{\theta^\alpha}(\varphi,\epsilon\varphi_X,\ldots)=\;&\epsilon\big[\gamma^\delta_\alpha(\mathbf{U})J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)\big]_X\\&-\epsilon\big[\big(\gamma^\delta_\alpha(\mathbf{U})\big)_X J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)+\gamma^\delta_\alpha(\mathbf{U})\partial^{II}_X J^{(s)}_\delta(\varphi,\partial^I_X\varphi,\ldots)\big]\\&+\epsilon\frac{\partial}{\partial\theta^\alpha}\delta h^{(s)}(\varphi,\ldots)+O(\epsilon^2)\end{aligned} \tag{5.7}$$

where

$$\delta h^{(s)}(\varphi,\ldots)=\frac{\partial h^{(s)}}{\partial\varphi^i_x}\big(\varphi,\partial^I_X\varphi,\ldots\big)\partial^{II}_X\varphi^i+\frac{\partial h^{(s)}}{\partial\varphi^i_{xx}}\big(\varphi,\partial^I_X\varphi,\ldots\big)\big(\partial^I_X\partial^{II}_X+\partial^{II}_X\partial^I_X\big)\varphi^i+\cdots$$

and the functions $\boldsymbol{\varphi}(\boldsymbol{\theta},X)$ have the form (5.6).

Let us now come back to the submanifolds $\mathcal{M}_\epsilon[\mathbf{\Psi}_{(1)}]$ and consider the functions $\varphi_{[\mathbf{U},\boldsymbol{\theta}^*]}$ having the form (5.2). We can see that relations (5.7) can then be rewritten in the form

$$\begin{aligned} h^{(s)}_{\theta^\alpha}(\varphi, \epsilon\varphi_X, \ldots) = {} & \epsilon\big[\gamma^\delta_\alpha(\mathbf{U}) J^{(s)}_\delta(\varphi, \partial^I_X\varphi, \ldots)\big]_X \\ & - \epsilon\big[\big(\gamma^\delta_\alpha(\mathbf{U})\big)_X J^{(s)}_\delta\big(\varphi, \partial^I_X\varphi, \ldots\big) + \gamma^\delta_\alpha(\mathbf{U})\partial^{II}_X J^{(s)}_\delta\big(\varphi, \partial^I_X\varphi, \ldots\big)\big] \\ & + \epsilon\frac{\partial}{\partial\theta^\alpha}\delta\tilde{h}^{(s)}(\varphi, \ldots) + O(\epsilon^2) \end{aligned} \tag{5.8}$$

where

$$\delta\tilde{h}^{(s)} = \delta h^{(s)} + \frac{\partial h^{(s)}}{\partial\varphi^i}\Psi^i_{(1)} + \frac{\partial h^{(s)}}{\partial\varphi^i_x}\epsilon\Psi^i_{(1)X} + \cdots .$$

We can now write on $\mathcal{M}_\epsilon[\mathbf{\Psi}_{(1)}]$

$$\begin{aligned} W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, X) = {} & \frac{1}{\epsilon}\int_{-\infty}^{+\infty}\nu(X-W)h^{(s)}_{\theta^\alpha}(\varphi, W)\,\mathrm{d}W \\ = {} & \gamma^\delta_\alpha(\mathbf{U}(X))J^{(s)}_\delta\big(\varphi, \partial^I_X\varphi, \ldots\big) \\ & - \int_{-\infty}^{+\infty}\nu(X-W)\partial^{II}_W\big(\gamma^\delta_\alpha(\mathbf{U}(W))J^{(s)}_\delta\big(\varphi, \partial^I_W\varphi, \ldots\big)\big)\,\mathrm{d}W \\ & + \int_{-\infty}^{+\infty}\nu(X-W)\frac{\partial}{\partial\theta^\alpha}\delta\tilde{h}^{(s)}(\boldsymbol{\theta}, W)\,\mathrm{d}W + O(\epsilon) \end{aligned} \tag{5.9}$$

(we use here the operator ∂^{II}_W also as ∂_W for the functions $\gamma^\delta_\alpha(\mathbf{U})$ depending on $\mathbf{U}$ only and assume the normalization of $J^{(s)}_\delta(\varphi, \ldots)$ such that $J^{(s)}_\delta(\boldsymbol{\theta}, \pm\infty) = 0$ on $\mathcal{M}$).

We can see that the quantities $W^{(s)}_{\theta^\alpha}$ have the order $O(1)$ for $\epsilon \to 0$ and the fixed coordinates $\mathbf{U}(x)$, $\boldsymbol{\theta}_0(X)$ on $\mathcal{M}_\epsilon[\mathbf{\Psi}_{(1)}]$.

We evidently have also

$$\int_0^{2\pi}\cdots\int_0^{2\pi} W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, X)\frac{\mathrm{d}^m\theta}{(2\pi)^m} = 0. \tag{5.10}$$

Let us now consider in the main order of ϵ the arbitrary value of the form

$$\int_0^{2\pi}\cdots\int_0^{2\pi} V(\boldsymbol{\theta}, X)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

where $V(\boldsymbol{\theta}, X)$ is arbitrary smooth and periodic w.r.t. $\boldsymbol{\theta}$ function (we can have in particular $X = Y$).

We have

$$\begin{aligned} \int_0^{2\pi}\cdots\int_0^{2\pi} V(\boldsymbol{\theta}, X)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m} = {} & \gamma^\delta_\alpha(Y)\int_0^{2\pi}\cdots\int_0^{2\pi} V(\boldsymbol{\theta}, X)J^{(s)}_\delta(\boldsymbol{\theta}, Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ & - \int_0^{2\pi}\cdots\int_0^{2\pi} V(\boldsymbol{\theta}, X)\int_{-\infty}^{+\infty}\nu(Y-W) \\ & \times \partial^{II}_W\big(\gamma^\delta_\alpha(W)J^{(s)}_\delta(\varphi, \partial^I_W\varphi, \ldots)\big)\,\mathrm{d}W\,\frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ & + \int_0^{2\pi}\cdots\int_0^{2\pi} V(\boldsymbol{\theta}, X)\int_{-\infty}^{+\infty}\nu(Y-W)\frac{\partial}{\partial\theta^\alpha} \\ & \times \delta\tilde{h}^{(s)}(\varphi(\boldsymbol{\theta}, W), \ldots)\,\mathrm{d}W\frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(\epsilon). \end{aligned} \tag{5.11}$$

The expressions $J_{\delta}^{(s)}(\varphi(\boldsymbol{\theta}, W), \ldots)$ and $\delta h^{(s)}(\varphi(\boldsymbol{\theta}, W), \ldots)$ are rapidly oscillating functions of W due to the fast change of the phase according to (5.2). It is not difficult to show that in the main order of ϵ expression (5.11) is given by the independent integration w.r.t. θ at the points X and W integrated then w.r.t. W for smooth generic $\mathbf{S}(W)$. We can see then that the third term in (5.11) disappears in fact in the main order of ϵ. After that we can also replace in the main order of ϵ the integration w.r.t. $\boldsymbol{\theta}$ just by averaging on the family Λ in the first two terms of (5.11) since the $\epsilon\boldsymbol{\Psi}_{(1)}$-corrections give there just the values of order $O(\epsilon)$. We can then write on $\mathcal{M}_{\epsilon}[\boldsymbol{\Psi}_{(1)}]$ in the main order of ϵ

$$\begin{aligned}\int_0^{2\pi} \cdots \int_0^{2\pi} V(\boldsymbol{\theta}, X) W_{\theta^{\alpha}}^{(s)}(\boldsymbol{\theta}, Y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} &= \gamma_{\alpha}^{\delta}(Y) \langle V(\boldsymbol{\theta}, X) J_{\delta}^{(s)}(\boldsymbol{\theta}, Y)\rangle \\ &\quad - \langle V(\boldsymbol{\theta}, X)\rangle \int_{-\infty}^{+\infty} \nu(Y-W) \partial_W \big(\gamma_{\alpha}^{\delta}(W) \langle J_{\delta}^{(s)}(\boldsymbol{\theta}, W)\rangle\big)\, \mathrm{d}W + o(1) \\ &= \gamma_{\alpha}^{\delta}(Y) \big[\langle V(\boldsymbol{\theta}, X) J_{\delta}^{(s)}(\boldsymbol{\theta}, Y)\rangle - \langle V(\boldsymbol{\theta}, X)\rangle \langle J_{\delta}^{(s)}(\boldsymbol{\theta}, Y)\rangle\big] + o(1).\end{aligned} \tag{5.12}$$

We can also write the following relation

$$\int_0^{2\pi} \cdots \int_0^{2\pi} V(\boldsymbol{\theta}, X) W_{\theta^{\alpha}}^{(s)}(\boldsymbol{\theta}, \pm\infty) \frac{\mathrm{d}^m\theta}{(2\pi)^m} = o(1)$$

for $\epsilon \to 0$ which follows from the formula (5.12) when we use $J_{\delta}^{(s)}(\boldsymbol{\theta}, \pm\infty) = 0$ on $\mathcal{M}$.

Looking now at the expression for $\Omega_{\nu\mu}^{1}(X, Y)$ we can see that all the terms containing values like $W_{\theta^{\alpha}}^{(s)}(\boldsymbol{\theta}, \pm\infty)$ can actually be omitted in the main $(1/\epsilon)$ order of $\Omega_{\nu\mu}^{1}(X, Y)$ according to the remark above.

Using the formula (5.4) we can write now

$$\begin{aligned}&-\frac{1}{\epsilon^2} \int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X) \nu(X-Z) \sum_{k\geqslant 0} \varphi_{\theta^{\alpha}}^{i}(\boldsymbol{\theta}, Z) \epsilon^k \omega_{ij}^{(k)}(\varphi(\boldsymbol{\theta}, Z), \ldots) \\ &\qquad \times \varphi_{\theta^{\beta}, kZ}^{j}(\boldsymbol{\theta}, Z) \nu(Z-Y) \frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\, \mathrm{d}Z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ &\qquad - \frac{1}{\epsilon^2} \int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X) \nu(X-Z) \\ &\qquad \times \sum_{s=1}^{g} e_s \big[h_{\theta^{\beta}}^{(s)}(\boldsymbol{\theta}, Z) T_{\alpha}^{(s)}(\boldsymbol{\theta}, Z) - h_{\theta^{\alpha}}^{(s)}(\boldsymbol{\theta}, Z) T_{\beta}^{(s)}(\boldsymbol{\theta}, Z) \\ &\qquad + \epsilon T_{\alpha, Z}^{(s)}(\boldsymbol{\theta}, Z) T_{\beta}^{(s)}(\boldsymbol{\theta}, Z)\big] \nu(Z-Y) \frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\, \mathrm{d}Z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ &\quad = -\frac{1}{\epsilon} \int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X) \nu(X-Z) [A_{\alpha\beta}(\varphi(\boldsymbol{\theta}, Z), \ldots)]_Z \\ &\qquad \times \nu(Z-Y) \frac{\partial k^{\beta}}{\partial U^{\mu}}(Y)\, \mathrm{d}Z \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ &\quad = -\frac{1}{\epsilon} \int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X) A_{\alpha\beta}(\varphi(\boldsymbol{\theta}, Z), \ldots) \nu(X-Y) \frac{\partial k^{\beta}}{\partial U^{\mu}}(Y) \frac{\mathrm{d}^m\theta}{(2\pi)^m} \\ &\qquad + \frac{1}{\epsilon} \int \frac{\partial k^{\alpha}}{\partial U^{\nu}}(X) \nu(X-Y) A_{\alpha\beta}(\varphi(\boldsymbol{\theta}, Y), \ldots) \frac{\partial k^{\beta}}{\partial U^{\mu}}(Y) \frac{\mathrm{d}^m\theta}{(2\pi)^m}.\end{aligned}$$

We can also use the identity

$$W^{(s)}_{\theta^\alpha,Z}(\boldsymbol{\theta},Z)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Z)=\tfrac{1}{2}\big(W^{(s)}_{\theta^\alpha}W^{(s)}_{\theta^\beta}\big)_Z+\tfrac{1}{2}\big(W^{(s)}_{Z}W^{(s)}_{\theta^\beta}\big)_{\theta^\alpha}-\tfrac{1}{2}\big(W^{(s)}_{Z}W^{(s)}_{\theta^\alpha}\big)_{\theta^\beta}.$$

Using now (5.5) and all the remarks above we can write (after some calculation)

$$\begin{aligned}
\Omega^1_{\nu\mu}(X,Y)=&-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)A_{\alpha\beta}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\ldots)\nu(X-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&+\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\sum_{s=1}^{g}e_sT^{(s)}_\alpha(\boldsymbol{\theta},X)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},X)\nu(X-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\sum_{s=1}^{g}e_s\frac{1}{2}W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},X)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},X)\nu(X-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)A_{\beta\alpha}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&+\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\sum_{s=1}^{g}e_sT^{(s)}_\beta(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\sum_{s=1}^{g}e_s\frac{1}{2}W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&+\frac{1}{\epsilon}\int\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\sum_{k\geqslant0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\ldots)\epsilon^k\\
&\times\Phi^i_{\theta^\beta,kX}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\nu(X-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\\
&\times\sum_{k\geqslant0}(-1)^k\epsilon^k\big[\Phi^i_{\theta^\alpha}(\boldsymbol{\theta}+\ldots,\mathbf{U}(Y))\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\big]_{kY}\\
&\times\Phi^j_{U^\mu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Y))\frac{\mathrm{d}^m\theta}{(2\pi)^m}+\frac{1}{\epsilon}\int\sum_{s=1}^{g}e_s\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}\\
&\times\big(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},X)-T^{(s)}_\beta(\boldsymbol{\theta},X)\big)\nu(X-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&+\frac{1}{\epsilon}\int\sum_{s=1}^{g}e_s\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)-T^{(s)}_\alpha(\boldsymbol{\theta},Y)\big)\frac{\delta\hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta},Y)}\\
&\times\Phi^j_{U^\mu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Y))\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&+\frac{1}{\epsilon}\int\sum_{s=1}^{g}e_s\frac{\partial k^\alpha}{\partial U^\nu}(X)\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},X)-T^{(s)}_\alpha(\boldsymbol{\theta},X)\big)\nu(X-Y)\\
&\times\big(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta},Y)-T^{(s)}_\beta(\boldsymbol{\theta},Y)\big)\frac{\partial k^\beta}{\partial U^\mu}(Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\\
&-\frac{1}{\epsilon}\int\sum_{s=1}^{g}e_s\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}\nu(X-Y)
\end{aligned}$$

$$\times \left(W^{(s)}_{\theta^\beta}(\boldsymbol{\theta}, Y) - T^{(s)}_\beta(\boldsymbol{\theta}, Y)\right) \frac{\partial k^\beta}{\partial U^\mu}(Y) \frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

$$- \frac{1}{\epsilon} \int \sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X) \left(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, X) - T^{(s)}_\alpha(\boldsymbol{\theta}, X)\right) \nu(X - Y)$$

$$\times \frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}, Y)} \Phi^j_{U^\mu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(Y)) \frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

$$+ \frac{1}{\epsilon} \int \sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(X)) \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, X)} \nu(X - Y)$$

$$\times \frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}, Y)} \Phi^j_{U^\mu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(Y)) \frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(1).$$

We will now investigate the weak limit $\Omega^{1(wl)}_{\nu\mu}(X, Y)$ of the form $\Omega^1_{\nu\mu}(X, Y)$, i.e. the limit in the sense of the integrals

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \xi^\nu(X) \Omega^1_{\nu\mu}(X, Y) \eta^\mu(Y) \, \mathrm{d}X \, \mathrm{d}Y$$

for fixed (smooth) $\xi^\nu(X)$ and $\eta^\mu(Y)$.

We will use first the formulae (5.9) for the values like $W^{(s)}_{\theta^\alpha}$, $W^{(s)}_{\theta^\beta}$ in the expression above. It is easy to see then that $\Omega^1_{\nu\mu}(X, Y)$ actually contains just the terms of order $1/\epsilon$ in the main part.

We note after this that the integration w.r.t. θ in the last four terms can be done independently at the points X and Y in the weak limit for the rapidly oscillating functions of X and Y in full analogy with the remark before the formula (5.12). Using then the formula (5.10) we see that values like $W^{(s)}_{\theta^\alpha}$, $W^{(s)}_{\theta^\beta}$ can actually be omitted in the order $1/\epsilon$ for the weak limit of the last four terms of $\Omega^1_{\nu\mu}(X, Y)$. We can also replace in the same terms the integration w.r.t. $\boldsymbol{\theta}$ just by averaging on the quasiperiodic solutions in the main $(1/\epsilon)$ order of ϵ.

It is not difficult to also prove the formula

$$\frac{\partial \langle h^{(s)} \rangle}{\partial U^\nu}(X) = \left\langle \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, X)} \Phi^i_{U^\nu}(\boldsymbol{\theta}, X) \right\rangle + \frac{\partial k^\alpha}{\partial U^\nu}(X) \langle T^{(s)}_\alpha(\boldsymbol{\theta}, X) \rangle \tag{5.13}$$

according to the definition (4.7) of the functions $T^{(s)}_\alpha$.

Using the formula (5.13) and the remarks above we can see then that the last four terms in the expression for $\Omega^1_{\nu\mu}(X, Y)$ give the terms

$$\frac{1}{\epsilon} \sum_{s=1}^{g} e_s \frac{\partial \langle h^{(s)} \rangle}{\partial U^\nu}(X) \nu(X - Y) \frac{\partial \langle h^{(s)} \rangle}{\partial U^\mu}(Y) + \frac{o(1)}{\epsilon}$$

for $\Omega^{1(wl)}_{\nu\mu}(X, Y)$.

If we now introduce the functions

$$\tau_{\beta\nu} = \frac{\partial k^\alpha}{\partial U^\nu} \left[-\langle A_{\alpha\beta} \rangle + \sum_{s=1}^{g} e_s \left\langle T^{(s)}_\alpha W^{(s)}_{\theta^\beta} \right\rangle - \frac{1}{2} \sum_{s=1}^{g} e_s \left\langle W^{(s)}_{\theta^\alpha} W^{(s)}_{\theta^\beta} \right\rangle \right]$$
$$+ \left\langle \Phi^i_{U^\nu} \sum_{k \geqslant 0} \omega^{(k)}_{ij}(\boldsymbol{\varphi}, \ldots) \varphi^j_{\theta^\beta, kx} \right\rangle + \left\langle \Phi^i_{U^\nu} \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \left(W^{(s)}_{\theta^\beta} - T^{(s)}_\beta\right) \right\rangle$$

and use the formulae (5.12) we can see that the form $\Omega^{1(wl)}_{\nu\mu}(X,Y)$ can be written as

$$\Omega^{1(wl)}_{\nu\mu}(X,Y)=\frac{1}{\epsilon}\sum_{\alpha=1}^{m}\left(\frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\nu(X-Y)\tau_{\alpha\mu}(Y)+\tau_{\alpha\nu}(X)\nu(X-Y)\frac{\partial k^{\alpha}}{\partial U^{\mu}}(Y)\right)$$
$$+\frac{1}{\epsilon}\sum_{s=1}^{g}e_s\frac{\partial\langle h^{(s)}\rangle}{\partial U^{\nu}}(X)\nu(X-Y)\frac{\partial\langle h^{(s)}\rangle}{\partial U^{\mu}}(Y)+\frac{o(1)}{\epsilon}$$

where the values $\tau_{\alpha\nu}(\mathbf{U})$ are given by the formulae (5.3) for the values $\partial I_{\alpha}/\partial U^{\nu}$.

Let us now prove that $\tau_{\alpha\nu}(\mathbf{U})$ can in fact be represented as the derivatives $\partial I_{\alpha}/\partial U^{\nu}$ for some functions $I_{\alpha}(\mathbf{U})$. We will assume as we stated already that the gradients $\mathrm{d}k^1,\ldots,\mathrm{d}k^m$ are linearly independent on $\mathcal{M}^N$. From the closeness of the form Ω^{rest} it follows that the form $\Omega^{1(wl)}_{\nu\mu}(X,Y)$ is also closed on $\mathcal{M}$. It is easy to see that the part

$$\frac{1}{\epsilon}\sum_{s=1}^{g}e_s\frac{\partial\langle h^{(s)}\rangle}{\partial U^{\nu}}(X)\nu(X-Y)\frac{\partial\langle h^{(s)}\rangle}{\partial U^{\mu}}(Y)$$

is closed according to theorem 2. We then get that the form

$$\frac{1}{\epsilon}\sum_{\alpha=1}^{m}\left(\frac{\partial k^{\alpha}}{\partial U^{\nu}}(X)\nu(X-Y)\tau_{\alpha\mu}(Y)+\tau_{\alpha\nu}(X)\nu(X-Y)\frac{\partial k^{\alpha}}{\partial U^{\mu}}(Y)\right)$$

should also be closed on $\mathcal{M}$. Using theorem 2 it is not difficult to see then that we should have $\tau_{\alpha\nu}(\mathbf{U})=\partial I_{\alpha}/\partial U^{\nu}$ for some functions $I_{\alpha}(\mathbf{U})$ in this case.

(II) We have $\Omega^2_{\nu\alpha}(X,Y)=-\Omega^3_{\alpha\nu}(Y,X)$ and

$$\Omega^2_{\nu\alpha}(X,Y)=\int\left(-\frac{1}{\epsilon}\frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Z)\varphi^i_{\theta^{\gamma}}(\boldsymbol{\theta},Z)+\delta(X-Z)\Phi^i_{U^{\nu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Z))\right)$$
$$\times\hat{\Omega}_{ij}(\boldsymbol{\theta},\boldsymbol{\theta}',Z,W)\varphi^j_{\theta^{\alpha}}(\boldsymbol{\theta}',W)\delta(W-Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\frac{\mathrm{d}^m\theta'}{(2\pi)^m}\,\mathrm{d}Z\,\mathrm{d}W.$$

It is easy to see that we can omit all the terms of order $O(1)$ in this expression keeping in mind the statement of the theorem. In particular, we can omit the differentiation of the function $\delta(W-Y)$ in the local part and write

$$\Omega^2_{\nu\alpha}(X,Y)=-\frac{1}{\epsilon}\int\frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Y)\varphi^i_{\theta^{\gamma}}(\boldsymbol{\theta},Y)$$
$$\times\sum_{k\geqslant0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\epsilon^k\varphi^j_{\theta^{\alpha},kY}(\boldsymbol{\theta},Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$-\int\sum_{s=1}^{g}e_s\frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Z)\big(W^{(s)}_{\theta^{\gamma}Z}(\boldsymbol{\theta},Z)-T^{(s)}_{\gamma,Z}(\boldsymbol{\theta},Z)\big)$$
$$\times\nu(Z-Y)\big(W^{(s)}_{\theta^{\alpha}Y}(\boldsymbol{\theta},Y)-T^{(s)}_{\alpha,Y}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Z$$
$$+\int\sum_{s=1}^{g}e_s\Phi^i_{U^{\nu}}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}\nu(X-Y)$$
$$\times\big(W^{(s)}_{\theta^{\alpha}Y}(\boldsymbol{\theta},Y)-T^{(s)}_{\alpha,Y}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}+O(1)$$
$$=-\int\frac{1}{\epsilon}\frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Y)\varphi^i_{\theta^{\gamma}}(\boldsymbol{\theta},Y)$$

$$\times \sum_{k\geqslant 0} \omega_{ij}^{(k)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\epsilon^k \varphi^j_{\theta^\alpha,kY}(\boldsymbol{\theta},Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$-\frac{\partial}{\partial Y}\int \sum_{s=1}^{g} e_s \frac{\partial k^\gamma}{\partial U^\nu}(X)\nu(X-Z)\big(W^{(s)}_{\theta^\gamma Z}(\boldsymbol{\theta},Z) - T^{(s)}_{\gamma,Z}(\boldsymbol{\theta},Z)\big)$$
$$\times \nu(Z-Y)\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Z$$
$$-\int \sum_{s=1}^{g} e_s \frac{\partial k^\gamma}{\partial U^\nu}(X)\nu(X-Y)\big(W^{(s)}_{\theta^\gamma Y}(\boldsymbol{\theta},Y) - T^{(s)}_{\gamma,Y}(\boldsymbol{\theta},Y)\big)$$
$$\times \big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m} + \frac{\partial}{\partial Y}\int \sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))$$
$$\times \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}\nu(X-Y)\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$+\int \sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}$$
$$\times \big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\delta(X-Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(1).$$

Using the relations $W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)\sim O(1)$, $\epsilon\to 0$ we can now omit the last two terms. The second term can be rewritten in the form

$$-\frac{\partial}{\partial Y}\int \sum_{s=1}^{g} e_s \frac{\partial k^\gamma}{\partial U^\nu}(X)\left(W^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},X) - T^{(s)}_{\gamma}(\boldsymbol{\theta},X) - \frac{1}{2} - W^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},+\infty)\right)$$
$$\times \nu(X-Y)\big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$+\frac{\partial}{\partial Y}\int \sum_{s=1}^{g} e_s \frac{\partial k^\gamma}{\partial U^\nu}(X)\nu(X-Y)\big(W^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},Y) - T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)\big)$$
$$\times \big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y) - T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

and can be omitted by the same reason.

So we have

$$\Omega^2_{\nu\alpha}(X,Y) = -\frac{1}{\epsilon}\int \frac{\partial k^\gamma}{\partial U^\nu}(X)\nu(X-Y)\varphi^i_{\theta^\gamma}(\boldsymbol{\theta},Y)$$
$$\times \sum_{k\geqslant 0} \omega_{ij}^{(k)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\epsilon^k \varphi^j_{\theta^\alpha,kY}(\boldsymbol{\theta},Y)\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$-\int \sum_{s=1}^{g} e_s \frac{\partial k^\gamma}{\partial U^\nu}(X)\nu(X-Y)\left(\frac{1}{2}\big(W^{(s)}_{Y}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)\big)_{\theta^\gamma}\right.$$
$$-\frac{1}{2}\big(W^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{Y}(\boldsymbol{\theta},Y)\big)_{\theta^\alpha} + \frac{1}{2}\big(W^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)\big)_{Y}$$
$$-\big(T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)\big)_{Y} + \frac{1}{\epsilon}h^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)$$
$$\left.-\frac{1}{\epsilon}h^{(s)}_{\theta^\gamma}(\boldsymbol{\theta},Y)T^{(s)}_{\alpha}(\boldsymbol{\theta},Y) + T^{(s)}_{\gamma,Y}(\boldsymbol{\theta},Y)T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)\right)\frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(1). \qquad (5.14)$$

We can now omit the total derivatives w.r.t. θ^{γ} and θ^{α} in the second integral. The term

$$-\int\sum_{s=1}^{g} e_s \frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Y)\left[\frac{1}{2}\big(W^{(s)}_{\theta^{\gamma}}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\big)_Y - \big(T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\big)_Y\right]\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

can be written as

$$-\frac{\partial}{\partial Y}\int\sum_{s=1}^{g} e_s \frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Y) \times\left[\frac{1}{2}W^{(s)}_{\theta^{\gamma}}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y) - T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\right]\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$
$$-\int\sum_{s=1}^{g} e_s \frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\delta(X-Y) \times\left[\frac{1}{2}W^{(s)}_{\theta^{\gamma}}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y) - T^{(s)}_{\gamma}(\boldsymbol{\theta},Y)W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},Y)\right]\frac{\mathrm{d}^m\theta}{(2\pi)^m}$$

and has also the order $O(1)$ for $\epsilon\to 0$.

The rest of expression (5.14) can now be written according to (5.4) as

$$-\frac{1}{\epsilon}\int\frac{\partial k^{\gamma}}{\partial U^{\nu}}(X)\nu(X-Y)\left[\frac{\partial}{\partial\theta^{\beta}}Q^{\beta}_{\gamma\alpha}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)+\epsilon\frac{\partial}{\partial Y}A_{\gamma\alpha}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\right]\frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(1) = O(1).$$

So we get part (II) of the theorem.

(III) We have

$$\begin{aligned}
\Omega^4_{\alpha\beta}(X,Y) ={}& \int\varphi^i_{\theta^{\alpha}}(\boldsymbol{\theta},X)\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\cdots)\epsilon^k\varphi^j_{\theta^{\beta},kX}(\boldsymbol{\theta},X)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\delta(X-Y)\\
&+\epsilon\int\sum_{s=1}^{g}e_s\big(W^{(s)}_{\theta^{\alpha}X}(\boldsymbol{\theta},X)-T^{(s)}_{\alpha,X}(\boldsymbol{\theta},X)\big)\nu(X-Y)\\
&\times\big(W^{(s)}_{\theta^{\beta}Y}(\boldsymbol{\theta},Y)-T^{(s)}_{\beta,Y}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\delta(X-Y)+O(\epsilon)\\
={}& \int\varphi^i_{\theta^{\alpha}}(\boldsymbol{\theta},X)\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta},X),\ldots)\epsilon^k\varphi^j_{\theta^{\beta},kX}(\boldsymbol{\theta},X)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\delta(X-Y)\\
&+\epsilon\frac{\partial^2}{\partial X\partial Y}\int\sum_{s=1}^{g}e_s\big(W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},X)-T^{(s)}_{\alpha}(\boldsymbol{\theta},X)\big)\nu(X-Y)\\
&\times\big(W^{(s)}_{\theta^{\beta}}(\boldsymbol{\theta},Y)-T^{(s)}_{\beta}(\boldsymbol{\theta},Y)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}+\epsilon\int\sum_{s=1}^{g}e_s\big(W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},X)-T^{(s)}_{\alpha}(\boldsymbol{\theta},X)\big)\\
&\times\big(W^{(s)}_{\theta^{\beta}}(\boldsymbol{\theta},X)-T^{(s)}_{\beta}(\boldsymbol{\theta},X)\big)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\delta'(X-Y)\\
&+\epsilon\int\sum_{s=1}^{g}e_s\left(\frac{1}{2}\big(W^{(s)}_{X}(\boldsymbol{\theta},X)W^{(s)}_{\theta^{\beta}}(\boldsymbol{\theta},X)\big)_{\theta^{\alpha}}-\frac{1}{2}\big(W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},X)W^{(s)}_{X}(\boldsymbol{\theta},X)\big)\right)_{\theta^{\beta}}\\
&+\frac{1}{2}\big(W^{(s)}_{\theta^{\alpha}}(\boldsymbol{\theta},X)W^{(s)}_{\theta^{\beta}}(\boldsymbol{\theta},X)\big)_X-\big(T^{(s)}_{\alpha}(\boldsymbol{\theta},X)W^{(s)}_{\theta^{\beta}}(\boldsymbol{\theta},X)\big)_X
\end{aligned}$$

$$+\frac{1}{\epsilon}h^{(s)}_{\theta^\beta}(\boldsymbol{\theta},X)T^{(s)}_\alpha(\boldsymbol{\theta},X)-\frac{1}{\epsilon}h^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},X)T^{(s)}_\beta(\boldsymbol{\theta},X)$$
$$+T^{(s)}_{\alpha,X}(\boldsymbol{\theta},X)T^{(s)}_\beta(\boldsymbol{\theta},X))\frac{\mathrm{d}^m\theta}{(2\pi)^m}\delta(X-Y)+O(\epsilon).$$

Using the same arguments as before we can now write

$$\Omega^4_{\alpha\beta}(X,Y)=\int\left[\frac{\partial}{\partial\theta^\gamma}Q^\gamma_{\alpha\beta}(\boldsymbol{\theta},X)\delta(X-Y)+\epsilon\left(A_{\alpha\beta}(\boldsymbol{\theta},X)\right)_X\delta(X-Y)\right]\frac{\mathrm{d}^m\theta}{(2\pi)^m}+O(\epsilon).$$

So we get part (III) of the theorem. □

Definition 4. *We call the form*

$$\Omega^{av}_{\nu\mu}(X,Y)=\sum_{\alpha=1}^{m}\left(\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial I_\alpha}{\partial U^\mu}(Y)+\frac{\partial I_\alpha}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial k^\alpha}{\partial U^\mu}(Y)\right)$$
$$+\sum_{s=1}^{g}e_s\frac{\partial\langle h^{(s)}\rangle}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial\langle h^{(s)}\rangle}{\partial U^\mu}(Y) \qquad (5.15)$$

the averaging of the form (2.3) on the space of m-phase solutions of system (3.1).

We call the functions $I^\alpha(\mathbf{U})$ defined through the formulae (5.3) the action variables conjugated with the wave numbers $k^\alpha(\mathbf{U})$.

We will now prove that the symplectic structure (5.15) can be considered actually as the symplectic structure for the Whitham system (3.15) while the value $\int\langle h\rangle(X)\,\mathrm{d}X$ plays the role of the Hamiltonian function for this system. Let us prove here the following theorem:

Theorem 5. *If the functions*

$$\phi^i_{(1)}(\boldsymbol{\theta},X,T,\epsilon)=\Phi^i\left(\frac{\mathbf{S}(X,T)}{\epsilon}+\boldsymbol{\theta}^*(X,T)+\boldsymbol{\theta},\mathbf{U}(X,T)\right)+\epsilon\Psi^i_{(1)}\left(\frac{\mathbf{S}(X,T)}{\epsilon}+\boldsymbol{\theta},X,T\right)$$

satisfy the system (3.9) modulo the terms $O(\epsilon^2)$ then the following relation is true

$$\int_{-\infty}^{+\infty}\Omega^{av}_{\nu\mu}(X,Y)U^\mu_T(Y)\,\mathrm{d}Y=\frac{\partial\langle h\rangle}{\partial U^\nu}(X). \qquad (5.16)$$

Proof. Let us prove first that under the conditions of the theorem the following relations hold in the weak limit:

$$\int\left(-\frac{1}{\epsilon}\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\phi^i_{(1)\theta^\alpha}(\boldsymbol{\theta},Z,\epsilon)+\delta(X-Z)\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Z))\right)$$
$$\times\hat{\Omega}_{ij}[\phi_{(1)}](\boldsymbol{\theta},\boldsymbol{\theta}',Z,W)\big(\phi^j_{(1)\theta'^\beta}(\boldsymbol{\theta}',W,\epsilon)\big(S^\beta_T(W)+\epsilon\theta^{*\beta}_T(W)\big)$$
$$+\epsilon\Phi^j_{U^\mu}(\boldsymbol{\theta}'+\cdots,\mathbf{U}(W))U^\mu_T(W)\big)\,\mathrm{d}Z\,\mathrm{d}W\frac{\mathrm{d}^m\theta}{(2\pi)^m}\frac{\mathrm{d}^m\theta'}{(2\pi)^m}$$
$$=\int\left(-\frac{1}{\epsilon}\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\phi^i_{(1)\theta^\alpha}(\boldsymbol{\theta},Z,\epsilon)+\delta(X-Z)\Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(Z))\right)$$
$$\times\frac{\delta\hat{H}}{\delta\varphi^i(\boldsymbol{\theta},Z)}\big(\phi^i_{(1)}(\boldsymbol{\theta},Z,\epsilon),\ldots\big)\,\mathrm{d}Z\frac{\mathrm{d}^m\theta}{(2\pi)^m}+o(1) \qquad (5.17)$$

$\epsilon\to 0$.

It is easy to see that the expression

$$\phi^j_{(1)\theta'^\beta}(\boldsymbol{\theta}',W,\epsilon)\big(S^\beta_T(W)+\epsilon\theta^{*\beta}_T(W)\big)+\epsilon\Phi^j_{U^\mu}(\boldsymbol{\theta}'+\cdots,\mathbf{U}(W))U^\mu_T(W)$$

actually gives the value $\epsilon\phi^j_{(1)T}(\boldsymbol{\theta}',W,\epsilon)$ up to the terms of order $O(\epsilon^2)$.

We can then write

$$\phi^j_{(1)\theta'^\beta}(\boldsymbol{\theta}', W, \epsilon)\big(S^\beta_T(W) + \epsilon\theta^{*\beta}_T(W)\big) + \epsilon\Phi^j_{U^\mu}(\boldsymbol{\theta}' + \cdots, \mathbf{U}(W))U^\mu_T(W)$$
$$= Q^j(\phi_{(1)}, \epsilon\phi_{(1)W}, \ldots) + \epsilon^2 G^j(\boldsymbol{\theta}' + \cdots, W)$$

where $G^j(\boldsymbol{\theta}', W)$ are some local expressions of $\boldsymbol{\Phi}(\boldsymbol{\theta}', \mathbf{U}(W))$, $\boldsymbol{\Psi}_{(1)}(\boldsymbol{\theta}', W)$ and their derivatives.

Let us start with the nonlocal part of the form $\hat{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', Z, W)$. First we note that

$$\int \left(-\frac{1}{\epsilon}\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\phi^i_{(1)\theta^\alpha}(\boldsymbol{\theta}, Z, \epsilon) + \delta(X-Z)\Phi^i_{U^\nu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(Z))\right)$$
$$\times \frac{1}{\epsilon}\sum_{s=1}^{g} e_s \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, Z)}[\phi_{(1)}]\nu(Z-W)\delta(\boldsymbol{\theta} - \boldsymbol{\theta}')\frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}', W)}[\phi_{(1)}]$$
$$\times \epsilon^2 G^j(\boldsymbol{\theta}' + \cdots, W)\,\mathrm{d}Z\,\mathrm{d}W \frac{\mathrm{d}^m\theta}{(2\pi)^m}\frac{\mathrm{d}^m\theta'}{(2\pi)^m}$$
$$= \int \left(-\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\Psi^i_{(0)\theta^\alpha}(\boldsymbol{\theta} + \cdots, Z)\right)$$
$$\times \sum_{s=1}^{g} e_s \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, Z)}[\boldsymbol{\Psi}_{(0)}]\nu(Z-W)\frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}, W)}[\boldsymbol{\Psi}_{(0)}]$$
$$\times G^j(\boldsymbol{\theta} + \cdots, W)\,\mathrm{d}Z\,\mathrm{d}W \frac{\mathrm{d}^m\theta}{(2\pi)^m} + O(\epsilon).$$

Using the same arguments as before we note that the rapidly oscillating functions of Z and W should be averaged in the weak limit separately in the main order of ϵ (for generic $\mathbf{S}(Z)$, $\mathbf{S}(W)$) and besides that

$$\langle \Psi^i_{(0)\theta^\alpha}(\boldsymbol{\theta} + \cdots, Z)\frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, Z)}[\boldsymbol{\Psi}_{(0)}]\rangle = \left\langle h^{(s)}_{\theta^\alpha} - \epsilon\frac{\partial}{\partial X}T^{(s)}_\alpha\right\rangle = -\epsilon\frac{\partial}{\partial X}\langle T^{(s)}_\alpha\rangle = O(\epsilon).$$

We can then claim that the terms consisting of G^j can actually be omitted since they affect (5.17) neither in the nonlocal nor in the local parts of $\hat{\Omega}_{ij}$.

Let us now use the relations

$$\frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}', W)}\epsilon\varphi^j_T(\boldsymbol{\theta}', W) = \frac{\delta \hat{H}^{(s)}}{\delta\varphi^j(\boldsymbol{\theta}', W)}Q^j(\varphi, \epsilon\varphi_W, \ldots) \equiv \epsilon\partial_W \bar{J}^{(s)}(\varphi, \epsilon\varphi_W, \ldots)$$

which follows from (3.3) (where the functions $\bar{J}^{(s)}$ are in general different from $J^{(s)}$ introduced in (3.3)).

Using the identity

$$\sum_{k\geqslant 0}\omega^{(k)}_{ij}(\boldsymbol{\theta}, Z)\epsilon^k\frac{\partial^k}{\partial Z^k}Q^j(\boldsymbol{\theta}, Z) + \sum_{s=1}^{g} e_s\frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, Z)}\bar{J}^{(s)}(\boldsymbol{\theta}, Z) \equiv \frac{\delta \hat{H}}{\delta\varphi^i(\boldsymbol{\theta}, Z)}$$

(which is the definition of the symplectic structure of the system (3.1)) we get (5.17).

Now using the relation

$$S^\beta_T(W) = \int_{-\infty}^{+\infty}\nu(W-Y)\frac{\partial k^\beta}{\partial U^\mu}(Y)U^\mu_T(Y)\,\mathrm{d}Y$$

we can see that the left-hand side of (5.17) can be written as

$$\epsilon\int_{-\infty}^{+\infty}\Omega^1_{\nu\mu}(X, Y)U^\mu_T(Y)\,\mathrm{d}Y + \epsilon\int_{-\infty}^{+\infty}\Omega^2_{\nu\beta}(X, Y)\theta^{*\beta}_T(Y)\,\mathrm{d}Y$$

where $\Omega^1_{\nu\mu}(X,Y)$, $\Omega^2_{\nu\beta}(X,Y)$ are the parts of the restriction of the form $\hat{\Omega}_{ij}$ on the manifold $\mathcal{M}_\epsilon[\mathbf{\Psi}_{(1)}]$ introduced in theorem 4.

Using relation (4.6) and the integration by parts (w.r.t. Z) in the right-hand side of (5.17) we can see that the right-hand side of (5.17) can be written as

$$\frac{\partial k^\alpha}{\partial U^\nu}(X)\langle T_\alpha(\boldsymbol{\theta}, X)\rangle + \left\langle \frac{\delta \hat{H}}{\delta\varphi^i(\boldsymbol{\theta}, X)} \Phi^i_{U^\nu}(\boldsymbol{\theta}, X)\right\rangle + O(\epsilon)$$

where T_α is the analogue of the functions $T^{(s)}_\alpha$ for the functional $\hat{H}$. So we have that the right-hand side of (5.17) is equal to $\partial\langle h\rangle/\partial U^\nu(X) + O(\epsilon)$ according to (5.13).

If we now consider the weak limit of relation (5.17) and use the parts (I) and (II) of theorem 4 we get relation (5.16) in the main $(O(1))$ order of ϵ. □

As we already stated previously, we can consider the system (5.16) as the Whitham system for (3.1) in the generic situation.

6. The weakly nonlocal 1-forms and the averaging of the weakly nonlocal Lagrangian functions

Let us consider now the 1-forms $\omega_i[\varphi](x)$ on the space of functions $\varphi^i(x)$, $i = 1, \ldots, n$ having the form

$$\omega_i[\varphi](x) = c_i(\varphi, \varphi_x, \ldots) - \frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-y) h^{(s)}(\varphi, \varphi_y, \cdots)\,\mathrm{d}y \qquad (6.1)$$

where $H^{(s)}[\varphi] = \int_{-\infty}^{+\infty} h^{(s)}(\varphi, \varphi_x, \ldots)\,\mathrm{d}x$.

We can see that the forms (6.1) have the purely local part and the nonlocal 'tail' of the fixed form which we will call weakly nonlocal in this situation. We will call the form $\omega_i[\varphi](x)$ purely local if it has the form

$$\omega_i[\varphi](x) = c_i(\varphi, \varphi_x, \ldots)$$

for some functions $c_i(\varphi, \varphi_x, \ldots)$.

We call the weakly nonlocal form (6.1) purely nonlocal if

$$\omega_i[\varphi](x) = -\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-y) h^{(s)}(\varphi, \varphi_y, \ldots)\,\mathrm{d}y.$$

The action of the forms $\omega_i[\varphi](x)$ on the 'tangent vectors' $\xi^i[\varphi](x)$ is defined in the natural way

$$(\boldsymbol{\omega}, \boldsymbol{\xi})[\varphi] = \int_{-\infty}^{+\infty} \omega_i[\varphi](x)\xi^i[\varphi](x)\,\mathrm{d}x.$$

The forms (6.1) are closely connected with the weakly nonlocal 2-forms (2.3). Namely, let us consider the external derivative of the form $\omega_i[\varphi](x)$:

$$[\mathrm{d}\boldsymbol{\omega}]_{ij}(x,y) = \frac{\delta\omega_j[\varphi](y)}{\delta\varphi^i(x)} - \frac{\delta\omega_i[\varphi](x)}{\delta\varphi^j(y)}.$$

Lemma 5. *The external derivative* $[\mathrm{d}\boldsymbol{\omega}]_{ij}(x,y)$ *is the closed 2-form having the form (2.3) with some local functions* $\omega^{(k)}_{ij}(\varphi, \varphi_x, \ldots)$.

Proof. First we note that the closeness of $d\omega$ is a trivial fact since $d\omega$ is exact. It is easy to see that the derivative of the local part of ω_i can be written as

$$\frac{\partial c_j}{\partial \varphi^i}(\varphi, \varphi_y, \ldots)\delta(y-x) + \frac{\partial c_j}{\partial \varphi^i_y}(\varphi, \varphi_y, \ldots)\delta'(y-x) + \cdots$$
$$-\frac{\partial c_i}{\partial \varphi^j}(\varphi, \varphi_x, \ldots)\delta(x-y) - \frac{\partial c_i}{\partial \varphi^j_x}(\varphi, \varphi_x, \ldots)\delta'(x-y) - \cdots$$

and is a purely local 2-form.

The derivative of the nonlocal part of ω_i can be written as

$$-\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta^2 H^{(s)}}{\delta\varphi^i(x)\delta\varphi^j(y)} \int_{-\infty}^{+\infty} \nu(y-z) h^{(s)}(\varphi, \varphi_z, \ldots)\, dz$$
$$-\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^j(y)} \int_{-\infty}^{+\infty} \nu(y-z) \frac{\delta h^{(s)}(\varphi, \varphi_z, \ldots)}{\delta\varphi^i(x)}\, dz$$
$$+\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta^2 H^{(s)}}{\delta\varphi^j(y)\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-z) h^{(s)}(\varphi, \varphi_z, \ldots)\, dz$$
$$+\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-z) \frac{\delta h^{(s)}(\varphi, \varphi_z, \ldots)}{\delta\varphi^j(y)}\, dz.$$

We have

$$\frac{\delta H^{(s)}}{\delta\varphi^i(x)} = \frac{\partial h^{(s)}}{\partial\varphi^i}(x) - \frac{\partial}{\partial x}\frac{\partial h^{(s)}}{\partial\varphi^i_x}(x) + \cdots$$

and

$$\frac{\delta^2 H^{(s)}}{\delta\varphi^i(x)\delta\varphi^j(y)} = \frac{\delta^2 H^{(s)}}{\delta\varphi^j(y)\delta\varphi^i(x)}$$

for smooth functions $h^{(s)}(\varphi, \varphi_x, \ldots)$.

We also have

$$\frac{\delta^2 H^{(s)}}{\delta\varphi^i(x)\delta\varphi^j(y)} = \frac{\delta^2 H^{(s)}}{\delta\varphi^j(y)\delta\varphi^i(x)} = \sum_{k\geqslant 0} A^{(s)k}_{ij}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y)$$

for some local functions $A^{(s)k}_{ij}(\varphi, \varphi_x, \ldots)$.

Using the formulae

$$\delta^{(k)}(x-y)\nu(y-z) = \delta^{(k)}(x-y)\nu(x-z) + \sum_{p=1}^{k} C^p_k \delta^{(k-p)}(x-y)\delta^{(p-1)}(x-z)$$

we can then write

$$-\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta^2 H^{(s)}}{\delta\varphi^i(x)\delta\varphi^j(y)} \int_{-\infty}^{+\infty} \nu(y-z) h^{(s)}(\varphi, \varphi_z, \ldots)\, dz$$
$$+\frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta^2 H^{(s)}}{\delta\varphi^j(y)\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-z) h^{(s)}(\varphi, \varphi_z, \ldots)\, dz$$

$$= -\frac{1}{2}\sum_{s=1}^{g} e_s \sum_{k\geqslant 1} A^{(s)k}_{ij}(\varphi, \varphi_x, \ldots)$$

$$\times \sum_{p=1}^{k} C^p_k (h^{(s)}(\varphi, \varphi_x, \ldots))_{(p-1)x} \delta^{(k-p)}(x-y)$$

which is a local expression.

Now we have

$$\int_{-\infty}^{+\infty} \nu(y-z) \frac{\delta h^{(s)}(\varphi, \varphi_z, \ldots)}{\delta \varphi^i(x)} \, \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} \nu(y-z) \left(\frac{\partial h^{(s)}}{\partial \varphi^i}(z)\delta(z-x) + \frac{\partial h^{(s)}}{\partial \varphi^i_z}(z)\delta'(z-x) + \ldots \right) \mathrm{d}z$$

$$= \sum_{p\geqslant 0} (-1)^p \left[\nu(y-x) \frac{\partial h^{(s)}}{\partial \varphi^i_{px}}(x) \right]_{px} = \nu(y-x) \frac{\delta H^{(s)}}{\delta \varphi^i(x)} + \text{(local part)}.$$

Also

$$\int_{-\infty}^{+\infty} \nu(x-z) \frac{\delta h^{(s)}(\varphi, \varphi_z, \ldots)}{\delta \varphi^j(y)} \, \mathrm{d}z = \nu(x-y) \frac{\delta H^{(s)}}{\delta \varphi^j(y)} + \text{(local part)}.$$

We have finally

$$[\mathrm{d}\boldsymbol{\omega}]_{ij}(x, y) = -\frac{1}{2} \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta \varphi^j(y)} \nu(y-x) \frac{\delta H^{(s)}}{\delta \varphi^i(x)}$$

$$+ \frac{1}{2} \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta \varphi^i(x)} \nu(x-y) \frac{\delta H^{(s)}}{\delta \varphi^j(y)} + \text{(local part)}$$

$$= \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta \varphi^i(x)} \nu(x-y) \frac{\delta H^{(s)}}{\delta \varphi^j(y)} + \text{(local part)}. \qquad \square$$

It is not difficult to prove also (using the analogous statement for purely local symplectic structures) that every closed 2-form (2.3) can be locally represented as the external derivative of some 1-form (6.1) on the space $\varphi(x)$.

We will give now the procedure for averaging 1-forms (6.1) connected with the averaging of the symplectic structures (2.3). Namely, we will assume now that the form $\Omega_{ij}(x, y)$ is represented as the external derivative of the form (6.1). The corresponding procedure of averaging of the form (6.1) should then give the weakly nonlocal 1-form of 'hydrodynamic type' which is connected with the form $\Omega^{av}_{\nu\mu}(X, Y)$ in the same way.

Definition 5. *We call the form $\omega_\nu[\mathbf{U}](X)$ on the space of functions $U^1(X), \ldots, U^N(X)$ the weakly nonlocal 1-form of hydrodynamic type if it has the form*

$$\omega_\nu[\mathbf{U}](X) = -\frac{1}{2} \sum_{s,p=1}^{M} \kappa_{sp} \frac{\partial f^{(s)}}{\partial U^\nu}(\mathbf{U}(X)) \int_{-\infty}^{+\infty} \nu(X-Y) f^{(p)}(\mathbf{U}(Y)) \, \mathrm{d}Y \qquad (6.2)$$

for some functions $f^{(s)}(\mathbf{U})$ and the quadratic form κ_{sp}.

It is not difficult to see that the form $\Omega_{\nu\mu}(X, Y)$ given by (1.17) is connected with (6.2) by the relation

$$\Omega_{\nu\mu}(X, Y) = [\mathrm{d}\omega]_{\nu\mu}(X, Y). \qquad (6.3)$$

As before, we introduce the extended space of functions $\varphi(\boldsymbol{\theta}, x)$ 2π-periodic w.r.t. each θ^{α}. After the change of coordinate $X = \epsilon x$ we can introduce the 1-form

$$\hat{\omega}_i(\boldsymbol{\theta}, X) = c_i(\varphi(\boldsymbol{\theta}, X), \epsilon\varphi_X(\boldsymbol{\theta}, X), \ldots) - \frac{1}{2\epsilon}\sum_{s=1}^{g} e_s \frac{\delta \hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta}, X)} \times \int_{-\infty}^{+\infty} \nu(X-Y) h^{(s)}(\varphi(\boldsymbol{\theta}, Y), \epsilon\varphi_Y(\boldsymbol{\theta}, Y), \ldots)\,\mathrm{d}Y \tag{6.4}$$

where $\hat{H}^{(s)} = \int_{-\infty}^{+\infty} h^{(s)}(\varphi(\boldsymbol{\theta}, X), \ldots)\,\mathrm{d}X$.

It is easy to see that the relation

$$\Omega_{ij}(x, y) = [\mathrm{d}\boldsymbol{\omega}]_{ij}(x, y)$$

gives

$$\hat{\Omega}_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, Y) = [\mathrm{d}\hat{\boldsymbol{\omega}}]_{ij}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, Y)$$

on the 'extended' functional space.

According to our previous approach we will investigate here the main term of the restriction of the 1-form $\hat{\omega}_i[\varphi](\boldsymbol{\theta}, X)$ on the submanifolds $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$ (in coordinates $(\mathbf{U}, \boldsymbol{\theta}_0^*)$) in the weak sense. Let us formulate here the corresponding theorem.

Theorem 6. *The restriction of the form $\hat{\omega}_i(\boldsymbol{\theta}, X)$ to any submanifold $\mathcal{M}_\epsilon[\boldsymbol{\Psi}_{(1)}]$ in coordinates $U^{\nu}(X)$, $\theta_0^{*\alpha}(X)$ can be written as*

$$\boldsymbol{\omega}^{\mathrm{rest}} = \int_{-\infty}^{+\infty} \omega_\nu^1(X)\delta U^\nu(X)\,\mathrm{d}X + \int_{-\infty}^{+\infty} \omega_\alpha^2(X)\delta\theta_0^{*\alpha}(X)\,\mathrm{d}X$$

where

(I) The form $\omega_\nu^1(X)$ can be written as

$$\omega_\nu^1(X) = -\frac{1}{\epsilon}\frac{\partial k^\alpha}{\partial U^\nu}(X)\int_{-\infty}^{+\infty} \nu(X-Y) I_\alpha(Y)\,\mathrm{d}Y - \frac{1}{2\epsilon}\sum_{s=1}^{g} e_s \frac{\partial\langle h^{(s)}\rangle}{\partial U^\nu}(X)\int_{-\infty}^{+\infty} \nu(X-Y)\langle h^{(s)}\rangle(Y)\,\mathrm{d}Y + \frac{o(1)}{\epsilon}$$

(summation over $\alpha = 1, \ldots, m$) where

$$I_\alpha(\mathbf{U}) = \left\langle c_i\varphi^i_{\theta^\alpha}\right\rangle + \frac{1}{2}\gamma_\alpha^\delta(\mathbf{U})\sum_{s=1}^{g} e_s\left[\left\langle h^{(s)} J_\delta^{(s)}\right\rangle - \langle h^{(s)}\rangle\left\langle J_\delta^{(s)}\right\rangle\right] - \frac{1}{2}\sum_{s=1}^{g} e_s\left\langle h^{(s)} T_\alpha^{(s)}\right\rangle \tag{6.5}$$

and the values $J_\delta^{(s)}(\varphi, \ldots)$, $\gamma_\alpha^\delta(\mathbf{U})$ and $T_\alpha^{(s)}(\varphi, \ldots)$ are introduced in (3.17), (3.19) and (4.7).

(II) The form $\omega_\alpha^2(X)$ has the order $O(1)$ for $\epsilon \to 0$.

Proof. (I) We have

$$\omega_\nu^1(X) = \int \left(\frac{1}{\epsilon}\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(Z-X)\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, Z) + \delta(Z-X)\Phi^i_{U^\nu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(Z))\right) \times \hat{\omega}_i(\boldsymbol{\theta}, Z)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Z$$

$$= -\frac{1}{\epsilon}\int \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)c_i(\varphi(\boldsymbol{\theta}, Z), \ldots)\varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, Z)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Z$$

$$+\frac{1}{2\epsilon^2}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},Z)\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},Z)}$$

$$\times\,\nu(Z-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y\,\mathrm{d}Z$$

$$-\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta}+\cdots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}$$

$$\times\,\nu(X-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y+O(1)$$

$$=\;-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)\langle c_i(\boldsymbol{\varphi}(\boldsymbol{\theta},Z),\ldots)\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},Z)\rangle\,\mathrm{d}Z$$

$$+\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)W^{(s)}_{\theta^\alpha Z}(\boldsymbol{\theta},Z)$$

$$\times\,\nu(Z-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y\,\mathrm{d}Z$$

$$-\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Z)T^{(s)}_{\alpha,Z}(\boldsymbol{\theta},Z)$$

$$\times\,\nu(Z-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y\,\mathrm{d}Z$$

$$-\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta}+\ldots,\mathbf{U}(X))\frac{\delta\hat{H}^{(s)}}{\delta\varphi^i(\boldsymbol{\theta},X)}$$

$$\times\,\nu(X-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\cdots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y+O(1).$$

Here $\langle\cdots\rangle$ means again the averaging on the family Λ and the functions $W^{(s)}$, $T^{(s)}_\alpha$ are the same as in (4.4), (4.7).

As in the proof of theorem 4 we can omit here also (by the same reason) the averaging with values like $W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},\pm\infty)$ in the main order of ϵ. We can write then

$$\omega^1_\nu(X)=-\frac{1}{\epsilon}\int\frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)\Big\langle c_i(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\varphi^i_{\theta^\alpha}(\boldsymbol{\theta},Y)\Big\rangle\,\mathrm{d}Y$$

$$+\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},X)\nu(X-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y$$

$$-\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta},Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y$$

$$-\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)T^{(s)}_{\alpha}(\boldsymbol{\theta},X)\nu(X-Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y$$

$$+\frac{1}{2\epsilon}\int\sum_{s=1}^{g} e_s \frac{\partial k^\alpha}{\partial U^\nu}(X)\nu(X-Y)T^{(s)}_{\alpha}(\boldsymbol{\theta},Y)h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta},Y),\ldots)\frac{\mathrm{d}^m\theta}{(2\pi)^m}\,\mathrm{d}Y$$

$$- \frac{1}{2\epsilon} \int \sum_{s=1}^{g} e_s \Phi^i_{U^\nu}(\boldsymbol{\theta} + \cdots, \mathbf{U}(X)) \frac{\delta \hat{H}^{(s)}}{\delta \varphi^i(\boldsymbol{\theta}, X)}$$
$$\times \nu(X - Y) h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, Y), \ldots) \frac{\mathrm{d}^m \theta}{(2\pi)^m} \, \mathrm{d}Y + O(1).$$

We can now use the same arguments as in the proof of theorem 4 and make in the main order of ϵ the independent integration w.r.t. θ of the rapidly oscillating functions depending on X and Y before the integration w.r.t. Y. We can then omit the second term of the expression above in the main order. Using also relations (5.12) and (5.13) we get the statement (I) of the theorem.

(II) We have

$$\omega^2_\alpha(X) = \int \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, X) \hat{\omega}_i(\boldsymbol{\theta}, X) \frac{\mathrm{d}^m \theta}{(2\pi)^m}$$
$$= \int c_i(\boldsymbol{\varphi}(\boldsymbol{\theta}, X), \ldots) \varphi^i_{\theta^\alpha}(\boldsymbol{\theta}, X) \frac{\mathrm{d}^m \theta}{(2\pi)^m} - \frac{1}{2} \int \sum_{s=1}^{g} e_s \big(W^{(s)}_{\theta^\alpha X}(\boldsymbol{\theta}, X) - T^{(s)}_{\alpha, X}(\boldsymbol{\theta}, X) \big)$$
$$\times \nu(X - Y) h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, Y), \ldots) \frac{\mathrm{d}^m \theta}{(2\pi)^m} \, \mathrm{d}Y.$$

Using the identity

$$-\frac{1}{2} \int \sum_{s=1}^{g} e_s \big(W^{(s)}_{\theta^\alpha X}(\boldsymbol{\theta}, X) - T^{(s)}_{\alpha, X}(\boldsymbol{\theta}, X) \big) \nu(X - Y) h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, Y), \ldots) \frac{\mathrm{d}^m \theta}{(2\pi)^m} \, \mathrm{d}Y$$
$$= -\frac{1}{2} \frac{\partial}{\partial X} \int \sum_{s=1}^{g} e_s \big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, X) - T^{(s)}_{\alpha}(\boldsymbol{\theta}, X) \big)$$
$$\times \nu(X - Y) h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, Y), \ldots) \frac{\mathrm{d}^m \theta}{(2\pi)^m} \, \mathrm{d}Y$$
$$+ \frac{1}{2} \int \sum_{s=1}^{g} e_s \big(W^{(s)}_{\theta^\alpha}(\boldsymbol{\theta}, X) - T^{(s)}_{\alpha}(\boldsymbol{\theta}, X) \big) h^{(s)}(\boldsymbol{\varphi}(\boldsymbol{\theta}, X), \ldots) \frac{\mathrm{d}^m \theta}{(2\pi)^m}$$

we easily get part (II) of the theorem. □

Definition 6. *We call the 1-form*

$$\omega^{av}_\nu(X) = -\frac{\partial k^\alpha}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X - Y) I_\alpha(Y) \, \mathrm{d}Y$$
$$- \frac{1}{2} \sum_{s=1}^{g} e_s \frac{\partial \langle h^{(s)} \rangle}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X - Y) \langle h^{(s)} \rangle(Y) \, \mathrm{d}Y \qquad (6.6)$$

where $I_\alpha(\mathbf{U})$ are defined by the formula (6.5) the averaging of the 1-form (6.1) on the family of m-phase solutions of (3.1).

As follows from our construction we have the relation

$$\Omega^{av}_{\nu\mu}(X, Y) = [\mathrm{d}\boldsymbol{\omega}^{av}]_{\nu\mu}(X, Y)$$

for the forms (5.15) and (6.6).

Using remark (6.3) it is not difficult to prove also that the quantities (6.5) give the action variables defined in (5.3).

We can see that the formulae (6.6), (6.5) give another procedure for the averaging of 2-forms $\Omega_{ij}(x, y)$ represented in the form of the external derivatives of weakly nonlocal 1-forms $\omega_i(x)$.

We can also write the formal Lagrangian formalism for the Whitham equations in the form

$$\delta \int\int \left[\omega_\nu^{av}(X) U_T^\nu(X) - \langle h \rangle(\mathbf{U})\right] \mathrm{d}X\, \mathrm{d}T = 0$$

or using (6.6)

$$\delta \int\int \Bigg[k_T^\alpha(X) \nu(X - Y) I_\alpha(Y) + \frac{1}{2} \sum_{s=1}^{g} e_s \langle h^{(s)} \rangle_T(X) \nu(X - Y) \langle h^{(s)} \rangle(Y) + \langle h \rangle \Bigg] \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}T = 0. \qquad (6.7)$$

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